

GROUPS WHICH ARE ALMOST GROUPS OF LIE TYPE IN CHARACTERISTIC p

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ABSTRACT. For a prime p , a p -subgroup of a finite group G is said to be large if and only if $Q = F^*(N_G(Q))$ and, for all $1 \neq U \leq Z(Q)$, $N_G(U) \leq N_G(Q)$. In this article we determine those groups G which have a large subgroup and which in addition have a proper subgroup H containing a Sylow p -subgroup of G with $F^*(H)$ a group of Lie type in characteristic p and rank at least 2 (excluding $\mathrm{PSL}_3(p^a)$) and $C_H(z)$ soluble for some $z \in Z(S)$. This work is part of a project to determine the groups G which contain a large p -subgroup.

1. INTRODUCTION

When classifying groups of Lie type in characteristic p , p a prime, one usually tries to determine the parabolic subgroups, construct a chamber system from the parabolic subgroups and finally identify the groups via the classification of the corresponding buildings. The parabolic subgroups in a group of Lie type are examples of p -local subgroups where, in an arbitrary group G , a p -local subgroup is by definition the normalizer of a non-trivial p -subgroup of G . Hence the first step in a classification theorem whose target groups are predominantly groups of Lie type in characteristic p should be to determine the structure of the maximal p -local subgroups containing a fixed Sylow p -subgroup. This approach has been started in a paper by Meierfrankenfeld, Stellmacher and the second author [11] where groups with a large p -subgroup are studied. Here, given a group G , a p -subgroup Q of G is *large* if and only if

- (L1) $Q = F^*(N_G(Q))$; and
- (L2) if U is a non-trivial subgroup of $Z(Q)$, then $N_G(U) \leq N_G(Q)$.

We will frequently use the fact that condition (L1) is equivalent to $Q = O_p(N_G(Q))$ and $C_G(Q) \leq Q$.

A motivating observation is that most of the groups of the Lie type in characteristic p contain a *large* p -subgroup. For example, in $\mathrm{PSL}_n(p^a)$,

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with $n \geq 3$, conjugates of the radical subgroups of the point and hyperplane stabilizers are large. If we consider $\mathrm{PSp}_{2n}(p^a)$ with p odd and $n \geq 2$, then the large subgroups are the radical subgroup of the stabilizer of a point. The group $\mathrm{PSp}_{2n}(2^a)$, $n \geq 2$, has no large 2-subgroups as $\mathrm{PSp}_{2n}(2^a)$ is generated by the centralizers of a long and a short root element contained in the centre of a fixed Sylow 2-subgroup.

We say that a p -local subgroup M of a group G is of *characteristic p* provided $F^*(M) = O_p(M)$ and we say that G is of *parabolic characteristic p* provided all p -local subgroups of G which contain a Sylow p -subgroup are of characteristic p .

A particularly appealing consequence of a group G containing a large p -subgroup is that G then is a group of parabolic characteristic p (Lemma 2.1 (iv)). This means that the existence of a large p -subgroup in a simple group is an indication that the group may be a Lie type group defined in characteristic p .

The contributions in [11] begin the identification of those groups which contain a large p -subgroup Q for some prime p . If we consider groups G of Lie type and Lie rank at least two which contain a large subgroup Q and fix a Sylow p -subgroup S of G with $Q \leq S$, then there is a maximal p -local subgroup M containing S such that $Q \not\leq O_p(M)$. So to mimic this higher Lie rank assumption in [11] it is assumed that there is such an M in general. Then the aim of the work in [11] is to provide information about the structure of the maximal p -local subgroups of G which do not normalize Q .

Once this goal is achieved one possible plan is to proceed as follows. Let M be a maximal p -local subgroup of G with $M \not\leq N_G(Q)$ and $C \leq N_G(Q)$ be minimal such that $C \not\leq N_M(Q)$. Then set $H = \langle M, C \rangle$. Now in the typical case one is able to show that H is an automorphism group of a group of Lie type in characteristic p , using the approach described in the first paragraph of this introduction. In fact in most cases H is the target group. Hence the remaining difficulty is to prove that $H = G$.

This difficulty can often be overcome as follows. First show that $N_G(Q) \leq H$ and then using this demonstrate that $N_G(E) \leq H$ for all non-trivial p -subgroups E which are normal in some Sylow p -subgroup of H . Having achieved this, show that H is a strongly p -embedded subgroup of G and use this to reach the conclusion that $H = G$ with the help of [16].

When p is odd, A. Seidel in his PhD thesis [20] has shown that the first two steps can be taken whenever the Lie rank of H is at least 3 and $N_H(Q)$ is not soluble. In work in progress G. Pientka is tasked to prove the same result when $p = 2$. Hence the open question is: *what are*

the obstructions to taking the first two steps when $N_H(Q)$ is soluble? The purpose of this paper is to answer this question. It is remarkable how often this situation arises. Precisely we prove

Theorem 1.1. *Assume that p is a prime, G is a group, $H = N_G(F^*(H))$ contains a Sylow p -subgroup of G and $F^*(H)$ is a simple group of Lie type in characteristic p and rank at least two but that $F^*(H) \not\cong \mathrm{PSL}_3(p^a)$ when p is odd.*

Suppose that a large subgroup Q of G is contained in H and $C_H(z)$ is soluble for some p -central element z of G . Then one of the following holds:

- (i) $N_G(Q) = N_H(Q)$;
- (ii) $p = 2$ and $F^*(G) \cong \mathrm{Mat}(11), \mathrm{Mat}(23), \mathrm{G}_2(3)$ or $\mathrm{P}\Omega_8^+(3)$; or
- (iii) $p = 3$ and $F^*(G) \cong \mathrm{PSU}_6(2), \mathrm{F}_4(2), {}^2\mathrm{E}_6(2), \mathrm{McL}, \mathrm{Co}_2, \mathrm{M}(22), \mathrm{M}(23)$ or F_2 .

We remark that the proof of Theorem 1.1 does not require a hypothesis, such as the \mathcal{K} -group hypothesis, on the composition factors of proper subgroups.

Regarding the omitted cases when $G = \mathrm{PSL}_3(p^a)$ with p odd and $p^a > 13$ we expect that it can be shown that $N_G(Q) = N_H(Q)$ (see [12, Theorem 1.5] to see why this should be the case). However, in the case $p^a \leq 13$, there are serious problems as the configuration we are examining is close to a configuration in the O’Nan simple group when $p = 7$ and the Monster simple group when $p = 13$.

In Section 2, we present various preliminary results that will be used in the proof of the main theorems. In particular, we produce the (well-known) list of simple Lie type groups defined in characteristic p of rank at least two in which the centralizer of some p -central element (a non-trivial element contained in the centre of a Sylow p -subgroup) is soluble. It transpires that this can only occur when either the rank of H is two or when $p^a \in \{2, 3\}$. Section 2 also contains amalgam type characterizations of the simple groups $\mathrm{Mat}(22)$ and $\mathrm{Mat}(23)$.

The proof of Theorem 1.1 is presented in Sections 3 and 4 where we deal with the configurations which arise when $p = 2$ and $p = 3$ respectively. When $p = 2$, the most troublesome situation arises when $H = \mathrm{PSL}_3(2^n)$ with $n \geq 2$. This is the situation which ultimately leads to the group $\mathrm{Mat}(23)$ and is close to configurations which exist in other simple groups which, however, fail to have a large subgroup. A common feature in the analysis is that Q often turns out to be an extraspecial p -group. In this case the possibilities for the structure of $N_G(Q)/Q$ can be determined as the outer automorphism group of such a group is either an orthogonal group of the appropriate type if $p = 2$ and if

p is odd and Q has exponent p then it is a general symplectic group [6, 20.5]. The overall strategy of the proof is to determine the possible structure of $N_G(Q)$ (where Q is the large subgroup) and then once this is done use characterization theorems to identify the groups from either 2-local or 3-local information. As an illustrative example, consider the possibility that $H \cong \mathrm{PSL}_4(3)$ or $\mathrm{PSU}_4(3)$. In this case we show that Q is an extraspecial group of order 3^5 and then, using the subgroup structure of $\mathrm{Out}(Q) \cong \mathrm{GSp}_4(3)$, we show that $N_G(Q)/Q$ has restricted structure. We then further investigate the 3-local structure of G until we have a good approximation to the structure of $N_G(Q)/Q$ at which stage we cite the appropriate recognition theorems [13, 14, 17, 18].

Throughout this article we follow Atlas [4] notation for group extensions. Indeed the Atlas is a good source for readers unfamiliar with the subgroup structure of the small simple groups to extract various pieces of useful information about the groups we shall encounter. Our group theoretic notation is mostly standard and follows that in [8] for example. For a prime p , we say that a non-trivial element is p -central provided its centralizer contains a Sylow p -subgroup of G .

For odd primes p , the extraspecial groups of exponent p and order p^{2n+1} are denoted by p_+^{1+2n} . The extraspecial 2-groups of order 2^{2n+1} are denoted by 2_+^{1+2n} if the maximal elementary abelian subgroups have order 2^{1+n} and otherwise we write 2_-^{1+2n} . We expect our notation for specific groups is self-explanatory. For a subset X of a group G , X^G denotes that set of G -conjugates of X . Often we shall give suggestive descriptions of groups which indicate the isomorphism type of certain composition factors. We refer to such descriptions as the *shape* of a group. Groups of the same shape have normal series with isomorphic sections. We use the symbol \approx to indicate that two groups have the same shape.

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2. PRELIMINARY RESULTS

In this section we present various results that we need for the proof of Theorem 1.1. The basic lemma about large subgroups that we shall use (without further reference) is as follows.

Lemma 2.1. *Suppose that Q is a large p -subgroup of G and T is a non-trivial p -subgroup of G such that $N_G(T) \geq Q$. Then*

- (i) $N_G(Q)$ contains the normalizer of a Sylow p -subgroup of G .

- (ii) If $Q \leq T$, then $N_G(T) \leq N_G(Q)$.
- (iii) $F^*(N_G(T)) = O_p(N_G(T))$.
- (iv) G has parabolic characteristic p .

Proof. Suppose that $S \in \text{Syl}_p(G)$ and $Q \leq S$. Then, by property (L1), $Z(S) \leq Z(Q)$ and, by property (L2), $N_G(S)$ normalizes Q . So (i) holds.

Suppose that $x \in N_G(T)$ and $Q \leq T$. Then $Q^x \leq T \leq S$. Therefore $N_G(Q)$ and $N_G(Q^x)$ both contain $N_G(S)$ by (i). It follows from Sylow's Theorem that $N_G(Q) = N_G(Q^x)$ and then $Q = Q^x$ by (L1).

Suppose that $Q \leq N_G(T)$ and set $A = E(N_G(T))O_{p'}(N_G(T))$. Then Q normalizes A . Since A centralizes $C_T(Q) \leq Z(Q)$, (L2) implies that A normalizes Q . But then $[Q, A] \leq Q \cap A$ is a normal p -subgroup of A , whence $[Q, A] \leq Z(A)$ and $[Q, A] = [Q, A, A] = 1$. Thus $A \leq Q$ by (L1) and this means that $A = 1$. Therefore (iii) holds and (iv) is a direct consequence of (iii). \square

Lemma 2.2. *Suppose that p is a prime, G is a group, $S \in \text{Syl}_p(G)$ has order p^a and V is a faithful GF(p) G -module of dimension $2a$. If, for all $s \in S^\#$,*

$$C_V(S) = [V, S] = C_V(s) = [V, s]$$

has dimension a , then either $|\text{Syl}_p(G)| = 1$ or $|\text{Syl}_p(G)| = p^a + 1$.

Proof. See [12, Lemma 4.16]. \square

The following lemma, which we will use several times, is an easy consequence of the Three Subgroup Lemma.

Lemma 2.3. *Suppose that p is a prime, P is a p -group of nilpotency class at most 2 and that $\alpha \in \text{Aut}(P)$ has order coprime to p . If α centralizes a maximal abelian subgroup of P , then $\alpha = 1$.*

Proof. As α has order coprime to p , we have $P = [P, \alpha]C_P(\alpha)$. Suppose that $E \leq C_P(\alpha)$ is a maximal abelian subgroup of P . Then $Z(P) \leq E$ and so is centralized by α . Thus, as P has nilpotency class at most 2,

$$[[P, C_P(\alpha)], \alpha] \leq [Z(P), \alpha] = 1.$$

Because we also have $[[C_P(\alpha), \alpha], P] = 1$, the Three Subgroup Lemma yields $[[P, \alpha], C_P(\alpha)] = 1$. Since $C_P(E) = E$, we then have $[P, \alpha] \leq E \leq C_P(\alpha)$. Thus $P = [P, \alpha]C_P(\alpha) = C_P(\alpha)$ and this proves the result. \square

Lemma 2.4. *Suppose that H is a simple group of Lie type defined in characteristic p of rank at least 2. Assume that $C_H(z)$ is soluble for some p -central element of H . Then one of the following holds:*

- (i) $H \cong \text{PSL}_3(p^a)$ for some $a \geq 1$;

- (ii) $p = 2$ and $H \cong \mathrm{Sp}_6(2)$, $\mathrm{PSU}_4(2) \cong \mathrm{PSp}_4(3)$, $\mathrm{PSU}_5(2)$, $\mathrm{G}_2(2)' \cong \mathrm{PSU}_3(3)$, ${}^2\mathrm{F}_4(2)$, $\Omega_6^+(2) \cong \mathrm{SL}_4(2)$, $\Omega_8^+(2)$ or $\mathrm{Sp}_4(2^a)'$ for some $a \geq 1$; or
- (iii) $p = 3$ and $H \cong \mathrm{PSp}_4(3) \cong \mathrm{PSU}_4(2)$, $\mathrm{PSL}_4(3)$, $\mathrm{PSU}_4(3)$, $\Omega_7(3)$, $\mathrm{P}\Omega_8^+(3)$ or $\mathrm{G}_2(3^a)$ for some $a \geq 1$.

Proof. Let $S \in \mathrm{Syl}_p(H)$ and n represent the rank of H . Then either $Z(S)$ is a long root group or $H \cong \mathrm{Sp}_{2n}(2^a)'$, $\mathrm{F}_4(2^a)$ or $\mathrm{G}_2(3^a)$ for $a \geq 1$ and $Z(S)$ is the product of the root groups corresponding to the highest long root and the highest short root ([9, Theorem 3.3.1]). Using [9, Theorem 3.2.2] it is easy to see that if $z \in Z(S)$ is a long root element and if $p^a > 3$ and $n \geq 3$, then $C_H(z)$ is non-soluble.

Suppose that $n = 2$ and that $H \not\cong \mathrm{PSL}_3(p^a)$. Let $z \in Z(S)$ be a long root element. If $H \cong \mathrm{PSp}_4(p^a)$ with $p^a > 3$ and odd, then $C_H(z)$ is non-soluble. The groups $\mathrm{PSp}_4(3)$ and $\mathrm{PSp}_4(2^a)'$ are listed in (ii) and (iii). If $G \cong \mathrm{PSU}_4(p^a)$ or $\mathrm{PSU}_5(p^a)$, then $C_H(z)$ contains a section isomorphic to $\mathrm{PSL}_2(p^a)$ or $\mathrm{PSU}_3(p^a)$ respectively. Hence $C_H(z)$ is non-soluble if $a \geq 2$ or $p^a = 3$ and $H \cong \mathrm{PSU}_5(3)$. Thus $\mathrm{PSU}_4(2)$ and $\mathrm{PSU}_5(2)$ are included in (ii) and $\mathrm{PSU}_4(3)$ is listed in (iii). If $G \cong \mathrm{G}_2(p^a)'$, then $C_H(z)$ contains a section isomorphic to $\mathrm{PSL}_2(p^a)$ and so is non-soluble unless $p^a \in \{2, 3\}$. The non-root elements in $Z(S)$ when $H \cong \mathrm{G}_2(3^a)$ have centralizer contained in the normalizer of $N_H(S)$. So (ii) lists $\mathrm{G}_2(2)'$ and (iii) includes $\mathrm{G}_2(3^a)$ for all positive a . If $H \cong {}^2\mathrm{F}_4(2^{2a+1})$, then $C_H(z)$ contains a section isomorphic to ${}^2\mathrm{B}_2(2^{2a+1})$ and is thus non-soluble if $a > 1$ and ${}^2\mathrm{F}_4(2)'$ is itemized in (ii). This completes the analysis when $n = 2$.

So assume that $n \geq 3$ and $p \in \{2, 3\}$. If $Z(S)$ is not a root group then $p = 2$ so we consider this case first. If $H \cong \mathrm{F}_4(2^a)$, then $C_H(Z(S))$ contains a section isomorphic to $\mathrm{Sp}_4(2^a)'$ and so this group is not listed. Suppose that $H \cong \mathrm{Sp}_{2n}(2^a)$. Then $C_H(Z(S))$ contains a section isomorphic to $\mathrm{Sp}_{2(n-2)}(2^a)'$. This group is not soluble if $2^a > 2$ or $n > 3$. Hence $\mathrm{Sp}_6(2)$ is listed in (ii). We may now additionally assume that $Z(S)$ is a root group and $n \geq 3$.

If $n \geq 4$, then $C_H(z)$ is non-soluble (containing a section of Lie rank at least 2) or $H \cong \mathrm{P}\Omega_8^+(p)$ and these groups are included in (ii) and (iii). We now may assume that the rank of H is 3 and that $p^a \in \{2, 3\}$.

If $p = 3$ then we include $H \cong \mathrm{PSL}_4(3)$, $\Omega_7(3)$ in (iii) and if $p = 2$, we have placed $\mathrm{SL}_4(2)$ in (ii). When $H \cong \Omega_8^-(p)$, $C_H(z)$ contains a section isomorphic to $\mathrm{PSL}_2(p^2)$. The possibilities $H \cong \mathrm{PSU}_6(p)$ or $\mathrm{PSU}_7(p)$ have $C_H(z)$ non-soluble as it has a section isomorphic to $\mathrm{PSU}_4(p)$ or $\mathrm{PSU}_5(p)$ respectively. If $H \cong \mathrm{PSp}_6(3)$, then $C_H(z)$ contains a section

isomorphic to $\mathrm{PSp}_4(3)$. So these latter groups are not included in the conclusion of the lemma. \square

We shall need the following specific fact about the normalizer of an extraspecial 2-subgroup of $\mathrm{Sp}_8(3)$.

Lemma 2.5. *Suppose that $G \cong \mathrm{Sp}_8(3)$ and X is an extraspecial subgroup of G of order 2^7 . Then $N_G(X)/X \cong \Omega_6^-(2) \cong \mathrm{PSU}_4(2)$ and, in particular, $N_G(X)$ contains no elements which act as transvections on $X/Z(X)$.*

Proof. This follows from [10, Proposition 4.6.9]. \square

Lemma 2.6. *Suppose that $X \cong \Omega_6^-(2) \cong \mathrm{PSU}_4(2)$, V is the natural 6-dimensional $\mathrm{GF}(2)\Omega_6^-(2)$ and let q be the associated quadratic form. Suppose that $Y^* = \mathrm{O}_2^-(2) \wr \mathrm{Sym}(3)$ and $Y = Y^* \cap X \approx 3^3.\mathrm{Sym}(4)$. Assume that F_1 and F_2 are fours groups in Y which are not Y -conjugate and that $F_1 O_3(Y)$ is normal in Y .*

- (i) *For involutions $t \in X$, $\dim C_V(t) = 4$ and $[V, t]$ contains a singular vector.*
- (ii) *$\dim C_V(F_1) = \dim C_V(F_2) = 3$.*
- (iii) *$C_V(F_1)$ contains singular vectors.*
- (iv) *There exists $f \in F_2$ such that $[C_V(f), F_2]$ contains singular vectors.*

Proof. The first assertion is well-known see [17, Lemma 2.2]. Let

$$\{x_1, x_2, y_1, y_2, z_1, z_2\}$$

be basis for V such that $\langle x_1, x_2 \rangle$, $\langle y_1, y_2 \rangle$ and $\langle z_1, z_2 \rangle$ are of --type and pairwise perpendicular. Moreover assume that the decomposition is preserved by Y . Then with respect to this basis we may suppose that F_1 is generated by the matrices

$$a = \mathrm{diag} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

and

$$b = \mathrm{diag} \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)$$

Hence $[V, a] = \langle x_2, y_2 \rangle$ and $[V, b] = \langle y_2, z_2 \rangle$ which means that $[V, F_1] = C_V(F_1) = \langle x_2, y_2, z_2 \rangle$ and this is clearly an isotropic space. Furthermore, we have $q(x_2 + y_2) = q(x_2) + q(y_2) = 1 + 1 = 0$. Thus $x_2 + y_2$ is singular.

We can assume that F_2 is generated by a and

$$c = \text{diag} \left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right)$$

(as the elements in $O_6^-(2) \setminus \Omega_6^-(2)$ have odd dimensional commutator spaces on V) and so $[V, F_2] = \langle x_2, y_2, x_1 + y_1 \rangle$ has dimension 3 and $[C_V(a), F_2] = \langle x_2 + y_2 \rangle$ is singular. This completes the proof of the lemma. \square

Lemma 2.7. *Suppose that V is an orthogonal module over GF(2) of dimension 8 and plus type. Let $X \cong \text{Sp}_2(2)\wr\text{Sym}(3)$ acts on V such that the base group $B \cong \text{Sp}_2(2) \times \text{Sp}_2(2) \times \text{Sp}_2(2)$ acts irreducibly. Assume that the direct factors of B are B_1 , B_2 and B_3 with, for $1 \leq i \leq 3$, $B_i = \langle a_i, b_i \rangle$ where a_i is an involution and b_i has order 3. Then*

- (i) $[V, a_1]$ is totally singular of dimension 4.
- (ii) $[V, a_1 a_2]$ is totally singular of dimension 4.
- (iii) $[V, a_1 a_2 a_3]$ is totally isotropic of dimension 4 but is not totally singular.
- (iv) If $f \in X \setminus B$ has order 2 and commutes with B_3 , then $[V, f]$ has dimension 2.
- (v) b_1, b_2, b_3 and their inverse are the only elements of order 3 in B which act fixed point freely on V .

Proof. There is a unique 8-dimensional representation for B over GF(2) and this is obtained as a tensor product of the natural symplectic modules for B_1 , B_2 and B_3 . For $i = 1, 2, 3$, let V_i be a natural module for B_i with symplectic basis $\{e_i, f_i\}$ and associated symplectic form $(,)_i$. Set $V = V_1 \otimes V_2 \otimes V_3$. Then V supports a symmetric bilinear form

$$(,) = \prod_{i=1}^3 (,)_i$$

and an associated quadratic form q which is entirely defined by specifying that all the pure tensors are singular. In this way B embeds into $O_8^+(2)$. This form is preserved by X which also has a unique 8-dimensional irreducible representation (the faithful representation degrees are 6, 8, 12 and 16).

We may suppose that $a_i \in B_i$ centralizes e_i and sends f_i to $e_i + f_i$. Then

$$[V, a_1] = \langle e_1 \otimes x \otimes y \mid x \in \{e_2, f_2\}, y \in \{e_3, f_3\} \rangle$$

which is totally singular. Similarly

$$[V, a_1 a_2] = \langle e_1 \otimes e_2 \otimes y, e_1 \otimes f_2 \otimes y + f_1 \otimes e_2 \otimes y + f_1 \otimes f_2 \otimes y \mid y \in \{e_3, f_3\} \rangle$$

is also totally singular. Finally

$$\begin{aligned} [V, a_1 a_2 a_3] &= \langle e_1 \otimes e_2 \otimes e_3, f_1 \otimes e_2 \otimes e_3 + e_1 \otimes f_2 \otimes e_3, \\ &\quad e_1 \otimes e_2 \otimes f_3 + e_1 \otimes f_2 \otimes e_3, \\ &\quad e_1 \otimes e_2 \otimes f_3 + f_1 \otimes f_2 \otimes e_3 + e_1 \otimes f_2 \otimes f_3 + f_1 \otimes e_2 \otimes f_3 \rangle. \end{aligned}$$

We see that this space is isotropic but that

$$q(e_1 \otimes e_2 \otimes f_3 + f_1 \otimes f_2 \otimes e_3 + e_1 \otimes f_2 \otimes f_3 + f_1 \otimes e_2 \otimes f_3) = (e_1 \otimes e_2 \otimes f_3, f_1 \otimes f_2 \otimes e_3) = 1$$

which means that this space is not totally singular. This proves (i), (ii) and (iii). If f is as in part (iv), then

$$C_V(f) = \langle e_1 \otimes e_2 \otimes y, f_1 \otimes f_2 \otimes y, e_1 \otimes f_2 \otimes y + f_1 \otimes e_2 \otimes y \mid y \in \{e_3, f_3\} \rangle.$$

So (iv) holds.

For $i = 1, 2, 3$, let b_i be the elements of B_i which maps e_i to $e_i + f_i$ and f_i to e_i . Then we see that $b_1 b_2$ centralizes $e_1 \otimes f_2 \otimes f_3 + f_1 \otimes e_2 \otimes f_3$ and that $b_1 b_2 b_3$ centralizes

$$e_1 \otimes e_2 \otimes e_3 + (e_1 + f_1) \otimes (e_2 + f_2) \otimes (e_3 + f_3) + f_1 \otimes f_2 \otimes f_3.$$

It is also simple to see that $[V, b_1] = V$. Thus (v) holds. \square

To deal with the possibility that $H^* \cong {}^2F_4(2)'$ we will need the following facts about this group.

Lemma 2.8. *Suppose that $X \cong {}^2F_4(2)$ and that X^* is the derived subgroup of X . Let $T \in \text{Syl}_2(X)$, $Z = Z(T)$, $V = Z_2(T)$, $P = C_G(Z)$, $R = O_2(P)$ and $M = N_G(V)$. For a subgroup Y of X set $Y^* = Y \cap X^*$.*

- (i) *The 2-rank of X and the 2-rank of X^* are both 5.*
- (ii) *$M/O_2(M) \cong \text{SL}_2(2)$, V has order 4 and is a natural $M/O_2(M)$ -module.*
- (iii) *$W = Z_3(T)$ has order 8 and is elementary abelian. M normalizes $Z_3(T)$ and $M/C_M(Z_3(T)) \cong \text{Sym}(4)$. Furthermore $|R : C_R(W)| = 4$.*
- (iv) *$Z_4(T)$ has order 16.*
- (v) *$P/R \cong P^*/R^*$ is isomorphic to a Frobenius group of order 20.*
- (vi) *$Z = Z(R^*)$ has order 2, $Z_2(R) = Z_2(R^*)$ is elementary abelian of order 2^5 and $Z_2(R)/Z$ is a P -chief factor.*
- (vii) *$R/C_R(Z_2(R))$ is a P -chief factor.*
- (viii) *$C_R(Z_2(R))$ is an abelian group of order 2^6 and $\Omega_1(C_R(Z_2(R))) = Z_2(R)$.*

Proof. That the 2-rank of X and X^* is 5 is given in [9, Theorem 3.3.3].

We use the results and notation from [9] especially Corollary 2.4.6 and the passages on pages 101 and 102. Thus we have root groups X_1 to X_{16} with X_i of order 2 if i is even and cyclic of order 4 if i is odd. For odd i we define $Y_i = \Omega_1(X_i)$. The opposite root group of X_i is X_{i+8} for $1 \leq i \leq 8$. We have $T = \prod_{i=1}^8 X_i$, $P = \langle T, X_9 \rangle$, $M = \langle T, X_1 \rangle$, $R = \prod_{i=2}^8 X_i$ and $O_2(M) = \prod_{i=1}^7 X_i$. In addition $M_1 = \langle X_8, X_{16} \rangle \cong \mathrm{SL}_2(2)$ and $P_1 = \langle X_1, X_9 \rangle \cong {}^2\mathrm{B}_2(2)$ is the Frobenius group of order 20.

We calculate that $Z(T) = Y_5$, $V = Z_2(T) = Y_5Y_3$, $Z_3(T) = Y_5Y_3X_4$, $Z_4(T) = Y_5Y_3X_4X_6$ using [9, Theorem 2.4.5 (d) and Corollary 2.4.6]. Further, using the same results, we verify that V and W are normalized by M and so this proves (ii), (iii) and (iv).

Using the statement in [9, pages 101 and 102], we get $Z(R) = Y_5$, $Z_2(R) = Y_5Y_3X_4X_6Y_7$ is elementary abelian of order 2^5 . We have $C_R(Z_2(R)) = Z_2(R)X_5$ and so this is an abelian group of order 2^6 which is not elementary abelian as X_5 is cyclic of order 4. Also from [9, pages 101, 102] we have that $R/C_R(Z_2(R))$ and $Z_2(R)/Z(R)$ are P -chief factors. In particular, as $[R, R] = Z_2(R) \leq X^*$, $Z_2(R) = Z_2(R^*)$. Finally X^* contains the elements $x_1(1)x_3(1)$ and $y_9(1)$ by [9, Theorem 3.3.2] and modulo R these two elements generate the ${}^2\mathrm{B}_2(2)$. Thus $P^*/R^* \cong {}^2\mathrm{B}_2(2)$. This discussion demonstrates (v), (vi), (vii) and (viii). \square

We shall also need the two following elementary lemmas.

Lemma 2.9. *Suppose that $X \cong \mathrm{SU}_4(3)$, $T \in \mathrm{Syl}_3(X)$ and $M = N_X(Z(T))$. Then M acts irreducibly on $O_3(M)/Z(T)$.*

Proof. We know that $O_3(M)$ is extraspecial of order 3^5 and $M/O_3(M)$ acts faithfully on $O_3(M)/Z(T)$ and contains a subgroup isomorphic to $\mathrm{SL}_2(3)$ at index 2. Now we note that the diagonal subgroup of X is homocyclic of order 16, it follows that $M/O_3(M)$ is not isomorphic to $\mathrm{GL}_2(3)$ and hence the action of M on $O_3(M)$ must be irreducible. \square

The next lemma exhibits some exceptional behaviour of $\mathrm{SL}_4(3)$.

Lemma 2.10. *Suppose that $X \cong \mathrm{SL}_4(3)$, $T \in \mathrm{Syl}_3(X)$ and $M = N_X(Z(T))$. Then M contains exactly four normal subgroups of order 27. Two of them are elementary abelian and two of them are extraspecial.*

Proof. We note that as $C_X(Z(T))/O_3(M) \cong \mathrm{SL}_2(3)$, there are at most 4 normal subgroups of order 27 in M . The two elementary abelian subgroups are easy to see. Take $Z(T)$ to be generated by the matrix

with ones down the diagonal and ones in the bottom left corner, the two extraspecial normal subgroups of M are generated by

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}.$$

□

Lemma 2.11. *Suppose that G is a group and P, B and L are subgroups of G such that $P \cong \mathrm{PSL}_3(4)$, $P \cap B \cap L$ is a Borel subgroup of P and $P \cap B$ and $P \cap L$ are point and line stabilizers in P respectively. Assume that $B \approx 2^4 : \mathrm{Alt}(6)$, $L \approx 2^4 : \mathrm{Sym}(5)$ and $|L : L \cap B| = 5$. Then $\langle P, B, L \rangle \cong \mathrm{Mat}(22)$.*

Proof. We may as well suppose that $G = \langle P, B, L \rangle$. We consider the graph Γ which has vertex set $\mathcal{P} \cup \mathcal{B}$ where $\mathcal{P} = \{Pg \mid g \in G\}$ and $\mathcal{B} = \{Bg \mid g \in G\}$ and edge set consisting of just the pairs $\{Pg, Bh\}$ such that $Pg \cap Bh \neq \emptyset$. The group G acts on Γ by right multiplication and the kernel of the action is a normal subgroup of P contained in $P \cap B$. As P is a simple group this means that the action of G is faithful. Since $L = (P \cap L)(L \cap B)$ the stabilizer of the connected component containing P and B is $G = \langle P, B \rangle$ and therefore Γ is connected. If L normalizes P , then, as $B = \langle B \cap P, B \cap L \rangle$, B also normalizes P . However $B \cap P$ is not normal in B and so this is impossible. Furthermore, as P acts on \mathcal{P} , we have that $|\mathcal{P}| \geq 22$.

For $\alpha \in \Gamma$ we let $\Gamma(\alpha)$ denote the set of neighbours of α in Γ . The pointwise stabilizer in G of a subset Θ of Γ is written as G_Θ . Note that if $\alpha = P$, then $G_\alpha = P$ and if $\beta = B$ then $G_\beta = B$ and the other stabilizers are conjugates of these groups. Our first observation is obvious. Let $\alpha = P$ and $\beta = B$. Then $|\Gamma(\alpha)| = |P : P \cap B| = 21$ and $|\Gamma(\beta)| = |B : B \cap P| = 6$. Moreover G_α acts on $\Gamma(\alpha)$ as it acts on the points of the projective plane and G_β acts as $\mathrm{Alt}(6)$ on $\Gamma(\beta)$.

Now $G_{\alpha\beta} = P \cap B \approx 2^4 : \mathrm{Alt}(5)$ acts transitively $\Gamma(\beta) \setminus \{\alpha\}$ and $G_{\alpha\beta\gamma} \approx 2^4 : \mathrm{Alt}(4)$ for any $\gamma \in \Gamma(\beta) \setminus \{\alpha\}$. Let $x \in (L \cap B) \setminus P$ and set $\gamma = Px$. Then $Px \cap B = Px \cap Bx = (P \cap B)x$ is non-empty. Thus $\gamma \in \Gamma(\beta)$. Furthermore, $P \cap P^x = P \cap L$ as $P \cap L$ is normalized by x and $P^x \neq P$. Thus $G_{\alpha\gamma} = P \cap L$ and this group acts on $\Gamma(\gamma)$

preserving the sets $\Gamma(\alpha) \cap \Gamma(\gamma)$ and $\Gamma(\gamma) \setminus (\Gamma(\alpha) \cap \Gamma(\gamma))$. Because $P \cap L$ is a line stabilizer in $G_\gamma = P^x$, it has orbits of lengths 5 and 16 on $\Gamma(\gamma)$. Since $|G_{\alpha\gamma} : G_{\alpha\beta\gamma}| = |P \cap L : P \cap L \cap B| = 5$, we infer that $\Gamma(\alpha) \cap \Gamma(\gamma)$ has order 5 or 21. If the size is 21, then we have accounted for all the cosets of B and P in G and we have 22 cosets of P and 22 cosets of B which is impossible as B and P have different orders. Thus $|\Gamma(\alpha) \cap \Gamma(\gamma)| = 5$. Let $\theta \in \Gamma(\beta)$ have distance 3 from α . Then $|\theta^{G_{\alpha\gamma}}| = 16$, and $G_{\alpha\gamma\theta} \cong \text{Alt}(5)$ complements $O_2(G_{\alpha\gamma})$. Consequently it acts on $\Gamma(\theta)$ with one fixed point γ and an orbit of length 5. In particular $G_\alpha = P$ acts transitively on the set of vertices at distance 3 from α . Now consider the path (β, α, τ) where $\tau \in \Gamma(\alpha) \setminus \{\beta\}$. Then $G_{\beta\alpha\tau}$ is the intersection of two point stabilizers in P and has shape $2^4 : 3$ and G_β acts transitively on such paths. We make such a path (α, β, Bx) where $x \in (P \cap L) \setminus B$ and note that the stabilizer of α and Bx contains $(B \cap L) \cap (B \cap L)^x \approx 2^4 : \text{Sym}(3)$. It follows that $\Gamma(\beta) \cap \Gamma(Bx)$ contains at least 2 vertices. The group $2^4 : \text{Sym}(3)$ acts on $\Gamma(\beta)$ with orbits of length 2 and 4 and therefore, since Γ is connected, we infer that $|\Gamma(\beta) \cap \Gamma(Bx)| = 2$.

In particular, $|\Gamma(\theta) \cap \Gamma(\beta)| \neq 1$. Since $G_{\alpha\gamma\theta}$ acts on $\Gamma(\theta)$ with an orbit of length 1 and an orbit of length 5, we deduce that every neighbour of θ is incident to some vertex at distance 2 from α . In particular $|\mathcal{P}| \leq 22$ and $|\mathcal{B}| \leq 77$. Since $|\mathcal{P}| \geq 22$, we have equalities $|\mathcal{P}| = 22$ and $|\mathcal{B}| = 77$. The fact that P acts two transitively on the 21 points of the projective plane yields that G acts three transitively on \mathcal{P} . In particular, given any three members of \mathcal{P} we may map them to three neighbours of the coset B .

We now identify the members of \mathcal{B} with their neighbours in \mathcal{P} . Thus \mathcal{B} becomes a set of six element subsets of \mathcal{P} which we call blocks. Since G acts three transitively on \mathcal{P} we get that any three points are contained in a block. Suppose that β_1 and β_2 are blocks sharing a common point. Then, as we saw earlier, $|\Gamma(\beta_1) \cap \Gamma(\beta_2)| = 2$ which means that every subset of \mathcal{P} of size 3 is contained in exactly one block. Thus $(\mathcal{P}, \mathcal{B})$ is a Steiner triple system with parameters $(3, 6, 22)$. Such systems are uniquely determined by [21] and therefore G is isomorphic to a subgroup of $\text{Aut}(\text{Mat}(22))$. As $G = \langle P, B \rangle$, we see $G = G'$. So $G \cong \text{Mat}(22)$ and this completes the proof of the lemma. \square

The proof of the next lemma is very similar to the previous one and so the proof is somewhat abbreviated.

Lemma 2.12. *Suppose that G is a group and P, B and L are subgroups of G such that $P \cong \text{Mat}(22)$, $B \approx 2^4 : \text{Alt}(7)$, $L \approx 2^4 : (3 \times \text{Alt}(5)).2$,*

$B \cap P \approx 2^4 : \text{Alt}(6)$, $L \cap P \approx 2^4 : \text{Sym}(5)$ and $P \cap B \cap L \approx 2^4 : \text{Sym}(4)$. Then $\langle P, B, L \rangle \cong \text{Mat}(23)$.

Proof. We again suppose that $G = \langle P, B, L \rangle$ and consider the graph Γ which has vertex set $\mathcal{P} = \{Pg \mid g \in G\}$ and $\mathcal{B} = \{Bg \mid g \in G\}$ made into a graph as in Lemma 2.11. Again we have $L = (P \cap L)(L \cap B)$ and that L does not normalize P . In particular, we have Γ is connected and G acts faithfully on Γ . We also have that $|\mathcal{P}| \geq 23$. For $\alpha \in \mathcal{P}$ and $\beta \in \mathcal{B}$ we know $|\Gamma(\alpha)| = 77$ and $|\Gamma(\beta)| = 7$.

Suppose that $\alpha = P$ and $\beta = B$. Then $G_{\alpha\beta} = P \cap B$ and so $G_{\alpha\beta}$ acts transitively on $\Gamma(\beta) \setminus \{\alpha\}$. Let $\gamma \in \Gamma(\beta) \setminus \{\alpha\}$. Then $G_{\alpha\beta\gamma} \approx 2^4 : \text{Alt}(5)$. Now we let $x \in (B \cap L) \setminus P$ and note that $\gamma = Px \in \Gamma(\beta)$ and that $G_\alpha \cap G_\gamma = P \cap P^x \geq (L \cap P)' \cong 2^4 : \text{Alt}(5)$. Furthermore we have $G_{\alpha\beta\gamma} \cap (L \cap P)' \approx 2^4 : \text{Alt}(4)$ has index 2 in $P \cap B \cap L$. We see that $\langle G_{\alpha\beta\gamma}, (L \cap P)' \rangle \cong \text{PSL}_3(4)$ and thus $G_{\alpha\gamma} \cong \text{PSL}_3(4)$ and, in particular, we have $|\Gamma(\alpha) \cap \Gamma(\gamma)| = 21$. Let $\theta \in \Gamma(\gamma)$ have distance 3 from α in Γ . Then as in $\text{Mat}(22)$ the stabilizer of a point p has an orbit of length 56 on the blocks not containing p , we see that $G_{\alpha\gamma\theta} \cong \text{Alt}(6)$. In particular $G_{\alpha\gamma\theta}$ has orbits of length 1 and 6 on $\Gamma(\theta)$. Thus we only need to see that $|\Gamma(\theta) \cap \Gamma(\beta)| \geq 3$. We do this exactly as in the last lemma and in fact we get that $|\Gamma(\beta) \cap \Gamma(\theta)| = 3$. Thus $|\mathcal{P}| = 23$ and $|\mathcal{B}| = 253$. We now prove that $G \cong \text{Mat}(23)$ just as in Lemma 2.11. \square

We assume that identifications of simple groups by their 2-local structure are fairly well known. This is not be the case for 3-local identifications. So to make the paper self contained we now state those results which will be used in this paper to identify groups when $p = 3$ in the main theorem.

Lemma 2.13. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of order 3^{1+4} and $Z(F^*(H)) = Z$;
- (ii) H/Q contains a normal subgroup isomorphic to $Q_8 \times Q_8$; and
- (iii) Z is not weakly closed in a Sylow 3-subgroup S of G with respect to G ,

then either $F^(G) \cong \text{F}_4(2)$ or $F^*(G) \cong \text{PSU}_6(2)$.*

Proof. See [17, Theorem 1.3]. \square

Lemma 2.14. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of type 3_+^{1+4} and $Z(F^*(H)) = Z$;
- (ii) $F^*(H/Q) = O_2(H/Q)$ is extraspecial of type 2_-^{1+4} ;
- (iii) $H/O_{3,2}(H) \cong \text{Alt}(5)$; and

- (iv) Z is not weakly closed in a Sylow 3-subgroup S of G with respect to G ,

then $G \cong \text{Co}_2$.

Proof. This is the main theorem of [14]. \square

Lemma 2.15. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. Let further S be a Sylow 3-subgroup of H and J be some elementary abelian subgroup of S of order 3^4 . If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of type 3_+^{1+4} and $Z(F^*(H)) = Z$;
- (ii) $F^*(H/Q) \cong 2 \cdot \text{Alt}(5)$; and
- (iii) $J = F^*(N_G(J))$ and $O^{3'}(N_G(J)/J) \cong \text{Alt}(6)$.

then $F^*(G) \cong \text{McL}$.

Proof. This is from [13]. \square

Lemma 2.16. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of order 3^{1+6} and $Z(F^*(H)) = Z$; and
- (ii) $O_2(H/Q) \cong Q_8 \times Q_8 \times Q_8$;
- (iii) Z is not weakly closed in a Sylow 3-subgroup S of G with respect to G ,

then $F^*(G) \cong {}^2\text{E}_6(2)$.

Proof. This is [15, Theorem 1.3]. \square

Lemma 2.17. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of order 3^{1+6} and $Z(F^*(H)) = Z$;
- (ii) $O_2(H/Q)$ acts on Q/Z as a subgroup of order 2^7 of $Q_8 \times Q_8 \times Q_8$, which contains $Z(Q_8 \times Q_8 \times Q_8)$; and
- (iii) Z is not weakly closed in a Sylow 3-subgroup S of G with respect to G ,

then $F^*(G) \cong \text{M}(22)$.

Proof. This is [15, Theorem 1.4]. \square

Lemma 2.18. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of type 3_+^{1+8} and $Z(F^*(H)) = Z$;
- (ii) $F^*(H/Q) = O_2(H/Q)$ is extraspecial of type 2_-^{1+6} ;
- (iii) $H/O_{3,2}(H)$ is isomorphic to the centralizer of a 3-central element in $\text{PSp}_4(3) \cong \Omega_6^-(2)$; and

- (iv) Z is not weakly closed in a Sylow 3-subgroup S of G with respect to G ,

then G is isomorphic to $M(23)$.

Proof. This is [19, Theorem 1.3]. \square

Lemma 2.19. *Suppose that G is a finite group, $Z \leq G$ has order 3 and set $H = C_G(Z)$. If the following hold*

- (i) $Q = F^*(H)$ is extraspecial of type 3_+^{1+8} and $Z(F^*(H)) = Z$;
- (ii) $F^*(H/Q) = O_2(H/Q)$ is extraspecial of type 2_-^{1+6} ;
- (iii) $H/O_{3,2}(H) \cong \Omega_6^-(2)$; and
- (iv) Z is not weakly closed in a Sylow 3-subgroup S of G with respect to G ,

then $G \cong F_2$.

Proof. This is [19, Theorem 1.4]. \square

3. THE CONFIGURATIONS WITH $p = 2$

We first of all establish some notation that will be used in this section and in Section 4. Assume that G and H are as in the statement of the main theorem. Thus $F^*(H)$ is a simple group of Lie type in characteristic p and of rank at least 2. We set $H^* = F^*(H)$, let S be a Sylow p -subgroup of H , $S^* = S \cap H^*$ and z be a root element in $Z(S^*)$. Throughout we assume that $Q \leq S$ is a large subgroup and that $N_G(Q) > N_H(Q)$. Note that if z centralizes Q then $C_H(z)$ normalizes Q and so $Q \leq O_p(C_H(z))$.

In this section we assume in addition that $p = 2$. We work through the various possibilities for H^* provided by Lemma 2.4.

Lemma 3.1. *If $H^* \cong \Omega_n^+(2)$ for $n = 6, 8$, then $Q = O_2(C_{H^*}(z))$ and is extraspecial.*

Proof. Note that $z \in Z(Q)$ and so $Q \leq O_2(C_H(z))$. If $n = 8$, then $C_H(z)$ is a maximal subgroup of H , $O_2(C_H(z))$ is extraspecial and $C_H(z)/O_2(C_H(z))$ acts irreducibly on $O_2(C_H(z))/\langle z \rangle$. Hence we have $Q = O_2(C_G(z))$ and the result is true in this case.

So assume that $n = 6$. Then $H^* \cong \mathrm{SL}_4(2)$ and $C_{H^*}(z) \approx 2_+^{1+4}.\mathrm{SL}_2(2)$ is not a maximal subgroup of H^* . We explore the possibilities for Q under the assumption that it is not $O_2(C_{H^*}(z))$.

The structure of $C_{H^*}(z)$ yields that in $O_2(C_{H^*}(z))$ there are exactly three non-central proper normal subgroups of $C_{H^*}(z)$. They each have order 2^3 and are abelian. Two of these groups are normal in parabolic subgroups of H^* which have Levi factors isomorphic to $\mathrm{SL}_3(2)$. These subgroups cannot be candidates for Q as otherwise $N_G(Q) = N_H(Q)$.

Let R be the third normal subgroup of $O_2(C_{H^*}(z))$ of order 2^3 which is normalized by $C_{H^*}(z)$ and assume that $Q \cap O_2(C_{H^*}(z)) = R$.

Assume that Q is not abelian. Then, as Q is normalized by $C_{H^*}(z)$ and $C_{H^*}(z)$ acts irreducibly on $R/\langle z \rangle$, R has index 2 in Q and $Z(Q) = \Phi(Q) = Q' = \langle z \rangle$. But then Q is extraspecial of order 2^4 which is ridiculous. Thus Q is abelian. Since R contains an element which is not a transvection, after identifying $\mathrm{SL}_4(2)$ with $\mathrm{Alt}(8)$, we may assume that $(1, 2)(3, 4) \in R$. But $N_{H^*}(R) = C_{H^*}(z)$ and this group does not contain $C_{\mathrm{Alt}(8)}((1, 2)(3, 4)) \approx (2^2 \times \mathrm{Alt}(4)).2$. Hence $Q \cap O_2(C_{H^*}(z)) \neq R$. Since $Q \cap O_2(C_{H^*}(z)) \neq \langle z \rangle$, we now know that $Q > O_2(C_{H^*}(z))$. Therefore $H \cong \mathrm{Sym}(8)$ and $Q = O_2(C_H(z)) > O_2(C_{H^*}(z))$. Set $Q_1 = O_2(C_{H^*}(z))$. Then $[Q, Q_1]$ is normalized by $C_H(z)$ and consequently has order 2^3 . It follows that Q_1 is the preimage of the Thompson subgroup of $Q/\langle z \rangle$ and therefore Q_1 is a characteristic subgroup of Q . In particular, $N_G(Q) \leq N_G(Q_1)$ and $N_G(Q)$ acts on Q_1 and centralizes Q/Q_1 . Thus all the elements of odd order in $N_G(Q)$ which centralize Q_1 also centralize Q . So we see that $O^2(N_G(Q)/Q_1)$ acts faithfully on Q_1 . But then $N_G(Q_1)/Q_1$ embeds into $O_4^+(2) \cong \mathrm{Sym}(3) \wr 2$ as Q_1 is extraspecial of order 2^5 and $+$ -type. As Q/Q_1 is a subgroup of order two which is centralized by $N_G(Q)$, we get that $N_G(Q) \leq H$, which is a contradiction. Thus $Q = O_2(C_{H^*}(z))$ and is extraspecial as claimed. \square

Lemma 3.2. *If $H^* \cong \Omega_n^+(2)$ for $n \in \{6, 8\}$, then $n = 8$ and $H^* \cong \Omega_8^+(2)$.*

Proof. We suppose that $H^* \cong \Omega_6^+(2)$ and aim for a contradiction. By Lemma 3.1 $Q = O_2(C_{H^*}(z))$ is extraspecial of order 2^5 . Hence $N_G(Q)/Q$ is isomorphic to a subgroup of $O_4^+(2)$ and $|N_G(Q) : N_H(Q)| = 3$. Let $\rho \in C_H(z)$ be of order three and $\nu \in N_G(Q)$ with $\langle \nu, \rho \rangle$ elementary abelian of order 9 and ν and ρ conjugate in the $O_4^+(2)$. Let $x \in N_{H^*}(Q) \setminus Q$ be a 2-element. Then x normalizes the three elementary abelian subgroups of order 8 in Q on which ρ acts (note that ρ also normalizes two quaternion subgroups of order 8). Furthermore ν permutes the three ρ -invariant elementary abelian subgroups of order 8 transitively. This gives that $[\nu, x] \in Q$ and hence ν normalizes $S^* = Q\langle x \rangle$. As the Thompson subgroup E of $Q\langle x \rangle$ is elementary abelian of order 16, $\nu \in N_G(E)$. Furthermore $N_H(E)$ induces either $\mathrm{Sym}(3) \times \mathrm{Sym}(3)$ or $\mathrm{Sym}(3) \wr 2$ on E . As $N_H(E)$ contains a Sylow 2-subgroup of $N_G(E)$, $N_H(E)$ acts irreducibly on E and $N_G(E)/C_G(E)$ is isomorphic to a subgroup of $\mathrm{SL}_4(2)$, we easily see that $N_G(E) = C_G(E)N_H(E)$. Since G is of parabolic characteristic 2, we also have $C_G(E) = E$. But then

$$\nu \in N_G(E) = C_H(E) \leq H,$$

which is a contradiction. We conclude that $N_G(Q) \leq H$ which is impossible. Hence $n \neq 6$. \square

Proposition 3.3. *If $H^* \cong \Omega_8^+(2)$, then $F^*(G) \cong P\Omega_8^+(3)$.*

Proof. Suppose that $H^* \cong \Omega_8^+(2)$. Then $C_{H^*}(z)/O_2(C_H(z)) \cong \mathrm{SL}_2(2) \times \mathrm{SL}_2(2) \times \mathrm{SL}_2(2)$, $Q = O_2(C_H(z)) = O_2(C_{H^*}(z))$ is extraspecial of order 2^9 by Lemma 3.1 and $|S : Q| \leq 2^4$ as $\mathrm{Out}(H^*) \cong \mathrm{Sym}(3)$. We know that $Z_2(S) = Z_2(S^*)$ has order 4 and is normalized by the parabolic subgroup P of H^* corresponding to the middle node of the Dynkin diagram of G and we have

$$P = \langle Q^g \mid g \in G, z^g \in Z_2(S) \rangle S^*$$

is normalized by $N_G(S^*)$. Thus $N_{N_G(N_{H^*}(Q))}(S^*)$ normalizes $N_{H^*}(Q)$ and P . Since $H^* = \langle P, N_{H^*}(Q) \rangle$, $N_{N_G(N_{H^*}(Q))}(S^*) \leq H$. In particular, $N_G(Q)$ does not normalize $N_{H^*}(Q)$ for otherwise

$$N_G(Q) = N_{H^*}(Q)N_{N_G(Q)}(S^*) \leq H.$$

Let $\overline{X} = N_G(Q)/Q$ and $\overline{Y} = O_3(\overline{N_H(Q)})$. Then $\overline{N_{H^*}(Q)} = \overline{S^*Y}$.

We have that $\overline{S^*}$ is elementary abelian and by Lemma 2.7 (i), (ii) and (iii) there is a unique element e in $\overline{S^*}$ such that $[Q/\langle z \rangle, e]$ is not totally singular. Thus e is weakly closed in $\overline{S^*}$ with respect to \overline{X} . We furthermore note that e inverts \overline{Y} . Suppose that $S > S^*$. Then $\overline{S} \cong 2 \times \mathrm{Dih}(8)$. Lemma 2.7 (iv) yields an involution $f \in \overline{S} \setminus \overline{S^*}$ such that $|(Q/\langle z \rangle, f)| = 4$ and so f is not \overline{X} -conjugate to any element of $\overline{S^*}$ as all the non-trivial elements in this group have commutator of order 2^4 on $Q/\langle z \rangle$. Let \overline{F} be the elementary abelian group of order 8 in \overline{S} with $\overline{F} \neq \overline{S^*}$. Then there are at most two elements (and they are conjugate) in $\overline{F} \setminus \overline{S^*}$ which could be \overline{X} -conjugate to an element of $\overline{S^*} \cap \overline{F}$. It follows that the other two elements in $\overline{S^*} \cap \overline{F}$ are not fused to any element of \overline{F} . Thus $N_{\overline{X}}(\overline{F})$ normalizes $\overline{S^*} \cap \overline{F}$ and

$$N_{\overline{X}}(\overline{F}) = N_{\overline{X}}(\overline{S^*} \cap \overline{F})S.$$

If e is not weakly closed in \overline{S} with respect to \overline{X} , then there is a \overline{X} -conjugate e^x of e in $\overline{F} \setminus \overline{S^*}$. Hence \overline{F}^x and \overline{F} are both contained in the centralizer of e^x and, as \overline{F} is not \overline{X} -conjugate to $\overline{S^*}$, we conclude that there exists $y \in C_{\overline{X}}(e^x)$ such that $\overline{F}^{xy} = \overline{F}$. Thus e^x and e are conjugate in $N_{\overline{X}}(\overline{F}) = N_{\overline{X}}(\overline{S^*} \cap \overline{F})S$ which is impossible. Therefore e is weakly closed in \overline{S} with respect to \overline{X} . Application of the Z^* -theorem [7] now shows that

$$e \in Z^*(\overline{X}).$$

In particular

$$\overline{Y} = [\overline{Y}, e] \leq O_{2'}(\overline{X}).$$

Suppose that $\overline{Y} = O_{2'}(\overline{X})$. Then as, by Lemma 2.7 (v), \overline{Y} contains exactly six elements which act fixed point freely on $Q/\langle z \rangle$ and these elements form three blocks of size two, we have that $\overline{X}/\overline{Y}$ embeds into $2 \wr \text{Sym}(3)$. Since when $S > S^*$, we have $N_{\overline{X}}(\overline{F}) = N_{\overline{X}}(\overline{S^*} \cap \overline{F})S$ we see that in any case $\overline{S^*Y}$ is normal in \overline{X} . But then the Frattini Argument gives $\overline{X} = \overline{Y}N_{\overline{X}}(S^*) \leq H$ which is a contradiction. Therefore $\overline{Y} < O_{2'}(\overline{X})$. The structure of $\text{Out}(Q) \cong O_8^+(2)$, now shows that $O_{2'}(\overline{X})$ is elementary abelian of order 81. As there is just one class of elementary abelian groups of order 81 in $\text{Out}(Q)$ (the Sylow 3-subgroup is contained in $O_2^-(2) \wr \text{Sym}(4)$), the action of $O_{2'}(\overline{X})$ on Q is uniquely determined. So the preimage of $O_{2'}(\overline{X})$ is a central product of four groups K_1, K_2, K_3 and K_4 isomorphic to $\text{SL}_2(3)$ and these are the only subgroups of $C_{H^*}(z) \cong \text{SL}_2(3)$. In particular, $N_G(Q)$ contains a subnormal subgroup isomorphic to $\text{SL}_2(3)$. Now if S^* normalizes K_i , $1 \leq i \leq 4$, then each $a \in S^*$ would have elements of order 4 in their commutator contrary to Lemma 2.7(i). Hence we see that there is an $i \in \{1, 2, 3, 4\}$ and $s \in S^*$ such that $[K_i^s, K_i] = 1$. Therefore an application of Aschbacher's Classical Involution Theorem [1, Theorems 7 and 8] yields $F^*(G)$ is either $P\Omega_n^\pm(3)$ for $5 \leq n \leq 8$ or $\text{Sp}_6(2)$. As $|S^*| = 2^{12}$, we see that $F^*(G) \cong P\Omega_8^\pm(3)$. However, $P\Omega_8^-(3)$ does not contain a subgroup isomorphic to $\Omega_8^+(2)$ (as 5^2 divides the order of one but not the other). Hence we have $F^*(G) \cong P\Omega_8^+(3)$. \square

Lemma 3.4. *We have that $H^* \not\cong \text{PSU}_4(2)$.*

Proof. Suppose that $H^* \cong \text{PSU}_4(2)$. Then $C_H(z)$ is a maximal subgroup of H and $C_H(z)$ acts irreducibly on $O_2(C_H(z))/\langle z \rangle$. This means that $N_H(Q) = C_H(z)$ and $Q = O_2(C_H(z))$. Thus Q is extraspecial of order 2^5 and has outer automorphism group $O_4^+(2) \cong \text{Sym}(3) \wr 2$. Since $|C_H(z)/O_2(C_H(z))|$ is divisible by 9, $O_{2,3}(N_G(Q)) \leq H$ and so $N_G(Q) \leq H$. This proves the lemma. \square

Lemma 3.5. *We have that $H^* \not\cong \text{PSU}_5(2)$.*

Proof. Suppose that $H^* \cong \text{PSU}_5(2)$. Then again $C_H(z) = N_H(Q)$ and this time Q is extraspecial of order 2^7 with outer automorphism group $O_6^-(2)$. Since $N_G(Q) \neq N_H(Q)$ and, by [4], $N_H(Q)/Q \cong \text{GU}_3(2)$ is a maximal subgroup of $\Omega_6^-(2)$, we have $N_G(Q)/Q$ contains a subgroup isomorphic to $\Omega_6^-(2)$. This is ridiculous as $|S : Q| = |N_H(Q)/Q|_2 \leq 2^4$ and therefore the lemma is true. \square

Proposition 3.6. *If $H^* \cong \text{G}_2(2)'$, then $G \cong \text{G}_2(3)$.*

Proof. In this case $C_H(z)$ is a maximal subgroup of H and so $C_H(z) = N_H(Q)$. We first show that $H \cong G_2(2)$. Otherwise we have $H \cong \mathrm{PSU}_3(3) \cong G_2(2)'$ and then $O_2(C_H(Z(Q))) \cong 4*Q_8$. But then $O_{2,3}(N_G(Q)) \leq H$ and so $N_G(Q) \leq H$, a contradiction. So $H \cong G_2(2)$.

Now $Q \leq O_2(C_H(z))$ is normal in $C_H(z)$. If $Q \neq O_2(C_H(z))$, then, as the element τ of order 3 in $C_H(z)$ has $[O_2(C_H(z)), \tau] \cong Q_8$, we have $[O_2(C_H(z)), \tau] \leq Q < O_2(C_H(z))$. It follows that $Q \cong Q_8$ or $4 * Q_8$ and then we have the same contradiction as in the last paragraph. Hence $Q = O_2(C_H(z)) \cong 2_+^{1+4}$ and the outer automorphism group of Q is $O_4^+(2) \cong \mathrm{Sym}(3) \wr 2$. As $N_G(Q) \not\leq H$, we now get that $|N_G(Q) : N_H(Q)| = 3$ and so $N_G(Q)$ contains a normal subgroup of index two isomorphic to $\mathrm{SL}_2(3) * \mathrm{SL}_2(3)$. We now consider the other parabolic subgroup P of H containing S . We have that P has shape $((4 \times 4) : 2).\mathrm{Sym}(3)$, where the homocyclic subgroup of shape 4×4 is inverted in $O_2(P)$. Let $x \in P \setminus C_H(z)$ and consider the subgroup $E = Q \cap Q^x$. Since $z \neq z^x$ and $\langle z^x, z \rangle$ has order 4 and is contained in Q , we get that E is elementary abelian and, as $|Q : Q \cap O_2(P)| = 2$ and $|(Q \cap O_2(P))Q^x/Q^x| \leq 2$, we infer that E has order 8. Additionally, as P has two non-central chief factors in $O_2(P)$, we have $P/E \cong \mathrm{Sym}(4)$.

If G has a subgroup G_1 of index 2, then, as $N_G(Q)$ does not normalize a subgroup of index two in Q , we get that $Q \leq G_1$ and of course $H^* \leq G_1$. But then $H = H^*Q \leq G_1$, a contradiction, as H contains a Sylow 2-subgroup of G . Thus G does not have a subgroup of index 2. In particular, by the Thompson Transfer Lemma [8, Lemma 15.16], all the involutions of H are conjugate to involutions in H^* . As H^* has a unique conjugacy class of involutions, the same is true for G . Therefore all the involutions in E are G -conjugate. Let $t \in E \setminus H^*$. Then t^P has order 4 and $C_P(t)E/E \cong \mathrm{Sym}(3)$. Especially we have $E = \langle t \rangle [E, C_P(t)]$. It follows that $E \leq C_G(t)'$. As $C_G(t)/O_2(C_G(t)) \cong N_G(Q)/Q$ has Sylow 2-subgroups of order 2, we have $E \leq O_2(C_G(t))$. In particular, there are elements of $N_G(E)$ which induce transvections on E with commutator $\langle t \rangle$. Using $P/E \cong \mathrm{Sym}(4)$, we now have $N_G(E)/C_G(E) \cong \mathrm{SL}_3(2)$. Since $C_G(E) = C_{N_G(Q)}(E) = E$, we now have $N_G(E)/E \cong \mathrm{SL}_3(2)$. Finally applying [2] we get that $G \cong G_2(3)$. \square

Lemma 3.7. *We have $H^* \not\cong \mathrm{Sp}_6(2)$.*

Proof. Since $H^* = H$, we have that $Q \geq Z(S)$ and then Q is normalized by $\langle C_H(z) \mid z \in Z(S^\#) \rangle = H$. This shows that $H^* \not\cong \mathrm{Sp}_6(2)$. \square

Lemma 3.8. *We have $H^* \not\cong {}^2\mathrm{F}_4(2)'$.*

Proof. Suppose that $H^* \cong {}^2\mathrm{F}_4(2)'$. Then, by Lemma 2.8 (v) and (vi), $C_{H^*}(z)$ has shape $2^{1+4+4} \cdot {}^2\mathrm{B}_2(2)$. Set $R = O_2(C_H(z))$ and $R^* = R \cap$

H^* . Suppose that $Q \not\geq R^*$. Then, as Q admits ${}^2B_2(2)$ non-trivially Lemma 2.8 (vii) implies that $Q \cap R^* = Z_2(R^*)$ is elementary abelian. If $Q > Q \cap R^*$, we further infer from Lemma 2.8(viii) that $|Q| = 2^6$, Q is abelian but not elementary abelian and that $Q \cap R^* = \Omega_1(Q)$. Hence $Q \cap R^*$ is normalized by $N_G(Q)$ and $N_G(Q \cap R^*)/C_G(Q \cap R^*)$ embeds into $GL_5(2)$ and contains a subgroup $N_{H^*}(Q)/C_G(Q \cap R^*) \approx 2^4 \cdot {}^2B_2(2)$. Since $[O_2(C_{H^*}(z)), Q \cap R^*] = \langle z \rangle$, we see that $N_{H^*}(Q)$ contains all the transvections to the point $\langle z \rangle$. As $C_{H^*}(z)$ acts irreducibly on $O_2(C_{H^*}(z))/(Q \cap R^*)$, it follows that either $N_G(Q) = N_H(Q)$ or $N_G(Q)/C_S(Q \cap R^*)$ is isomorphic to $GL_5(2)$. The first is impossible by assumption and the second implies that $2^{12} \geq |S| \geq 2^{15}$ which is absurd. Hence $Q \geq O_2(C_{H^*}(z))$. Suppose that $Q = O_2(C_{H^*}(z))$ and $H > H^*$. Then, by Lemma 2.8 (vi) and (vii), $Q/Z_2(Q)$, $Z_2(Q)/Z(Q)$ and $Z(Q)$ are irreducible $N_G(Q)/Q$ -sections which are all centralized by $O_2(C_H(z))$ and this is impossible. Thus $Q = O_2(C_H(z))$.

In particular, $N_G(Q)/Q$ has cyclic Sylow 2-subgroups and consequently $N_G(Q)/Q$ has a normal 2-complement. Since $Z(Q) = \langle z \rangle$ and $Z_2(Q)$ is elementary abelian of order 2^5 , $N_G(Q)/C_G(Z_2(Q))$ embeds in to the parabolic subgroup of shape $2^4 : SL_4(2)$ in $SL_5(2)$. As $Q/C_Q(Z_2(Q))$ is elementary abelian of order 2^4 , we now have $N_G(Q)/Q$ is isomorphic to a subgroup of $SL_4(2)$ and therefore $O^2(N_G(Q)/Q)$ must be a cyclic group of order 15 and furthermore the centralizer of the element of order 3 in $N_G(Q)/Q$ is isomorphic to $3 \times \text{Dih}(10)$. Let τ be an element of order 3 in $N_G(Q)$, β be an element of order 5 which commutes with τ and $t \in S$ be a 2-element that centralizes τ and is not contained in Q . Note that as τ is contained in a cyclic group of order 15, $C_{O_2(C_{H^*}(z))}(\tau) = C_{O_2(C_{H^*}(z))}(\beta)$ has order 2. Set $L = \langle \tau, S \rangle$. Further assume that M is the parabolic subgroup of H containing S with $M \neq N_H(z)$. Set $K = \langle L, M \rangle$.

As $Z_2(Q)/Z(Q)$ is the 4-dimensional ${}^2B_2(2)$ -module and as t inverts β mod Q , $C_{Z_2(Q)/Z(Q)}(t)$ has order 2^2 and is normalized by S . Let W be the preimage of $C_{Z_2(Q)/Z(Q)}(t)$. Then W is elementary abelian of order 2^3 and τ acts non-trivially on $W/Z(Q)$. Furthermore L normalizes W and W corresponds to $Z_3(T)$ in Lemma 2.8. Hence exploiting Lemma 2.8 (iii) again we see that W is normalized by M and that M acts as $\text{Sym}(4)$ on W . Thus K normalizes W and $K/C_K(W) \cong SL_3(2)$. Observe that $C_K(W)$ centralizes $Z(Q)$ and hence normalizes Q as Q is large. Since $N_G(Q)/Q \approx 15 \cdot 4$, we infer that $C_K(W) = C_Q(W)$ has order 2^8 or 2^9 . Moreover, we remark that $C_{C_Q(W)}(\tau)$ has order at most 8. If $|C_K(W)| = 2^8$, then there are at most two 3-dimensional $K/C_K(W)$ -modules involved in $C_Q(W)$. Thus, in this case, $|C_{C_K(W)}(\tau)| \geq 2^{1+1+2} = 16$, which

is a contradiction. So we have $|C_K(W)| = 2^9$ and all the chief-factors for K in $C_K(W)$ are 3-dimensional $K/C_K(W)$ -modules. In particular $H > H^*$. By Lemma 2.8(iv), $|Z(S/W)| = 2$. Hence $C_K(W)/W$ is a non-split extension of two 3-dimensional modules. Choose $x \in Z_2(Q)$ so that x projects to a non-trivial element in $C_{Z_2(Q)/W}(S)$ and set $U = \langle x^K \rangle W$. We have that U/W is of order 2^3 . As K acts transitively on the vectors in the natural 3-dimensional $\mathrm{SL}_3(2)$ -module and x has order 2, we have that U has exponent 2 and so is elementary abelian. This contradicts Lemma 2.8 (i) and proves the lemma. \square

Proposition 3.9. *If $H^* \cong \mathrm{Sp}_4(2)'$, then $G \cong \mathrm{Mat}(11)$.*

Proof. If $H = H^* \cong \mathrm{Sp}_4(2)' = \mathrm{Alt}(6)$, then G has dihedral Sylow 2-subgroups of order 8. The candidates for Q are the two elementary abelian groups of order 4, the cyclic group of order 4 and the Sylow 2-subgroup itself. In each case we have $N_G(Q) \leq H$ and we are done.

Suppose that $|H : H^*| = 2$. Then $S \cong 2 \times \mathrm{Dih}(8)$, $\mathrm{Dih}(16)$ or $\mathrm{SDih}(16)$ corresponding to $H \cong \mathrm{Sym}(6)$, $\mathrm{PGL}_2(9)$ and $\mathrm{Mat}(10)$ respectively. Consider the first case. Then $|Z(S)| = 4$ and $Z(S) \leq Z(Q)$. But $H = \langle C_H(z) \mid z \in Z(S)^\sharp \rangle \leq N_G(Q)$, which is impossible.

If S is a dihedral group of order 16, then all the candidates for Q have 2-groups as their automorphism group. So again $N_G(Q) \leq H$, a contradiction. So suppose that S is semidihedral of order 16. Then the possibilities for Q are a quaternion group of order 8, a cyclic subgroup of order 8, a dihedral subgroup of order 8 and S itself. Since the only one of these groups with a non-trivial automorphism of odd order is the quaternion group, we infer that $C_G(z) \cong \mathrm{GL}_2(3)$ and consequently $G \cong \mathrm{Mat}(11)$ or $\mathrm{PSL}_3(3)$ by [5]. Since 5 divides the order of H but not the order of $\mathrm{PSL}_3(3)$, we conclude that $G \cong \mathrm{Mat}(11)$.

Suppose that $|H : H^*| = 4$. Let M be the subgroup of H with $M \cong \mathrm{Mat}(10)$. Then all the involutions in M are contained in H^* . Let $t \in H \setminus M$ be an involution such that $C_S(t) \cong \mathrm{Dih}(8) \times 2$ (so t is in the subgroup of H^* isomorphic to $\mathrm{Sym}(6)$). Since $Z(S)$ has order 2, we see that it is impossible for t to be conjugate to a 2-central involution and therefore G has a subgroup G^* of index 2 with $t \notin G^*$ by the Thompson Transfer Lemma [8, Theorem 15.16]. In particular G^* has dihedral or semidihedral Sylow 2-subgroups. If $Q \cap G^*$ is a large subgroup of G^* , we may apply induction and get that $G^* \cong \mathrm{Mat}(11)$ and obtain a contradiction as $\mathrm{Out}(\mathrm{Mat}(11))$ is trivial. Thus $Q \cap G^*$ is not a large subgroup of G^* . Certainly $Q \cap G^*$ is normalized by $N_G(Q) \cap G^*$ and $O^2(N_G(Q)) \leq G^*$. It follows that $Q \cap G^*$ is a quaternion group of order 8 and $|Z(Q \cap G^*)| = 2$. We now have that $C_G(Z(Q \cap G^*))$ normalizes

$Q \cap G^*$. In addition, $C_{G^*}(Q \cap G^*) \leq N_G(Q)$ and so $Q \cap G^*$ is a large subgroup of G^* and we have a contradiction. \square

Lemma 3.10. *We have that $H^* \not\cong \mathrm{Sp}_4(2^a)$ with $a \geq 2$.*

Proof. Suppose that in fact $H^* \cong \mathrm{Sp}_4(2^a)$ for some $a \geq 2$.

Let P_1 and P_2 denote the maximal parabolic subgroups of H^* containing S^* and define $E_i = O_2(P_i)$ for $i = 1, 2$. Thus E_1 and E_2 are elementary abelian 2-subgroups of S^* of order 2^{3a} , $S^* = E_1E_2$, $E_1 \cap E_2 = Z(S^*)$ and every element of order 2 in S^* is contained in $E_1 \cup E_2$. Furthermore, for $i = 1, 2$, we have $O^{2'}(P_i)/E_i \cong \mathrm{SL}_2(2^a)$ and E_i is the $\Omega_3(2^a)$ -module for $O^2(P_i)$.

If $Z(Q)$ contains a root elements of H^* , then we may suppose that $N_G(Q)$ contains P_1 from which it follows that $Q = E_1$. But then Q contain both long and short root elements and thus $N_G(Q)$ also contains P_2 which is a contradiction as $H = \langle P_1, P_2 \rangle$. It follows that $H > H^*$ and that Q contains an element which acts as a graph automorphism of H^* . Furthermore, $Z(Q)$ contains no elements which are conjugate to root elements of H^* .

Since Q is normal in S and contains an element α with $E_1^\alpha = E_2$ we have that $(Q \cap S^*)E_1 = [E_1, \alpha]E_1 = S^*$. Therefore

$$Z(S^*) = [E_1, S^*] = [E_1, Q \cap S^*] \leq Q$$

as E_1 is the orthogonal module for $O^{2'}(P_1)/E_1$ and $a \neq 1$. Thus we have $Z(S^*) = E_1 \cap E_2 \leq Q$. We claim that $E_1 \cap Q$ and $E_2 \cap Q$ are the unique elementary abelian subgroups of maximal rank in Q (they may be equal). Suppose that E is a further elementary abelian subgroup of Q . We have $E \cap S^* \leq E_1$ or $E \cap S^* \leq E_2$ as all the involutions of S^* are contained in $E_1 \cup E_2$. So we may assume that E is not contained in S^* . Then $C_{E_1 \cap E_2}(E)$ has index 2^a in $E_1 \cap E_2$, and so we see that $E \cap S^*$ has index at least 4 in $E_1 \cap Q$ or in $E_2 \cap Q$ and this proves our claim since then $|E| < |E_1 \cap Q|$. We now have that $E_1 \cap E_2 = E_1 \cap Q \cap E_2 \cap Q$ is a characteristic subgroup of Q and thus we have

$$N_G(Q) \leq N_G(E_1 \cap E_2).$$

Since $E_1 \cap E_2$ is normalized by S and G is of parabolic characteristic 2, we have

$$F^*(N_G(E_1 \cap E_2)) = O_2(N_G(E_1 \cap E_2)).$$

Now S^* is a Sylow 2-subgroup of $C_G(E_1 \cap E_2)$ and as $S^*/(E_1 \cap E_2)$ is abelian, we have $S^* = F^*(C_G(E_1 \cap E_2)) = O_2(C_G(E_1 \cap E_2))$. In particular, S^* is normalized by $N_G(Q)$ and so $S^* \leq Q = O_2(N_G(Q))$ and we conclude that $Q = S$ as S/S^* is a cyclic group which is generated by the graph automorphism [9, Theorem 2.5.12].

Since $N_G(Q) = N_G(S)$, $N_G(Q)$ permutes E_1 and E_2 , $N_G(Q)$ has a subgroup N_0 of index 2 which normalizes both E_1 and E_2 and furthermore $N_G(Q) = N_{N_G(Q)}(E_1)Q$. Let $Q_0 = Q \cap N_0$. Then Q_0 is normal in $N_G(Q)$.

We claim that N_0 normalizes H^* . We have $E_1 \cap E_2$ contains two root subgroups R_1 and R_2 and no involution of $(E_1 \cap E_2) \setminus (R_1 \cup R_2)$ is conjugate into either R_1 or R_2 , as such involutions are conjugate to involutions in $Z(Q)$ and the involutions in R_1 and R_2 are not. Since N_0 normalizes $E_1 \cap E_2$, N_0 also normalizes R_1 and R_2 (as $S \cap N_0$ does). It follows that $M = \langle N_0, P_1 \rangle$ acts on R_1 and we get $C_M(R_1)$ is a normal subgroup of M which contains $O^{2'}(P_1)$. Suppose that $C_M(E_1) \neq E_1$. Then $C_M(E_1) \cap S \leq C_S(E_1) = E_1$ and so we infer that $C_M(E_1) = E_1 J$ where J is a group of odd order. Since J normalizes $S^* = F^*(C_G(E_1 \cap E_2))$ which has class 2 and since E_1 is a maximal abelian subgroup of S^* , J centralizes S^* by Lemma 2.3. Therefore $J = 1$ and $C_G(E_1) = E_1$. As $S^* \in \text{Syl}_2(C_G(R_1))$, we may apply Lemma 2.2 to the action of $C_M(R_1)$ on E_1/R_1 to see that $O^{2'}(P_1)$ is a characteristic subgroup of $C_M(R_1)$. Thus N_0 normalizes $O^{2'}(P_1)$ and similarly N_0 normalizes $O^{2'}(P_2)$ and, as $H^* = \langle O^{2'}(P_1), O^{2'}(P_2) \rangle$, this proves our claim. Finally we now have that N_0 normalizes H as does Q and thus $N_G(Q) \leq H$ and we have our contradiction. \square

Lemma 3.11. *If $H^* \cong \text{PSL}_3(2^a)$, then $a > 1$.*

Proof. Suppose $n = 1$. Then either Q is elementary abelian of order 4, a dihedral group of order 8, a dihedral group of order 16, or a cyclic group. In all of these cases we have $N_G(Q) \leq H$, which is a contradiction. \square

Lemma 3.12. *Assume $H^* \cong \text{PSL}_3(2^a)$ and that one of the following holds:*

- (i) $a > 2$; or
- (ii) $O_2(N_G(Z(S^*))) \leq S^*$.

Then $F^(C_G(Z(S^*))) = S^*$.*

Proof. As $Z(S^*)$ is normal in S and G has parabolic characteristic 2, $F^*(N_G(Z(S^*))) = O_2(N_G(Z(S^*)))$.

We have that

$$[O_2(N_G(Z(S^*))), N_{H^*}(S^*)] \leq O_2(N_G(Z(S^*))) \cap H^* \leq S^*.$$

If $a > 2$, $\text{Out}(H^*)$ acts faithfully on $N_{H^*}(S^*)/S^*$ and so we conclude that $O_2(N_G(Z(S^*))) \leq S^*$. So we may assume that we have (ii) holds. But then

$$[O_2(N_G(Z(S^*))), S^*] \leq Z(S^*)$$

and so $S^* = O_2(N_G(Z(S^*)))$ as claimed. \square

Lemma 3.13. *Assume $H^* \cong \mathrm{PSL}_3(2^a)$. Let $S^* = E_1 E_2$ where E_1 and E_2 are the elementary abelian subgroups of order 2^{2a} in S^* . Then either $N_G(E_1) \not\leq H$ or $N_G(E_2) \not\leq H$.*

Proof. Assume that $N_G(E_1)$ and $N_G(E_2)$ are both contained in H . Then $Q \neq E_1$ and $Q \neq E_2$. By Lemma 3.11 $a > 1$. Set $U = Z(S^*) = E_1 \cap E_2$. We remark that every involution of S^* is contained in either E_1 or E_2 and that, for $i = 1, 2$, $O^2(N_{H^*}(E_i)/E_i) \cong \mathrm{SL}_2(2^a)$ with E_i a natural $\mathrm{SL}_2(2^a)$ -module.

First we show that

(3.13.1) $U \leq Q$.

If $Q \leq S^*$, then we simply have $U = Z(S^*) \leq C_G(Q) \leq Q$. If $Q \not\leq S^*$, then we argue that there is an element of $x \in [Q, S^*] \setminus E_1$ and note that for such elements we have $U = [x, E_1] \leq Q$. Thus in both cases $U \leq Q$. \blacksquare

(3.13.2) $N_G(Q) \leq N_G(U)$.

Aiming for a contradiction suppose that $N_G(U) \not\geq N_G(Q)$.

Assume first that $Q \not\leq H^*$. Suppose additionally that $a > 2$. If there is an element x of Q which induces a field automorphism of order 2 on H^* , then, for $i = 1, 2$, $[E_i, x]U \leq Q$ has order $2^{3a/2}$. Hence $|E_1 \cap Q| \geq 2^{3a/2} \leq |E_2 \cap Q|$. Let F be an elementary abelian subgroup of Q with $|F| \geq 2^{3a/2}$. Assume that $F \not\leq S^*$ and let $y \in F \setminus S^*$. Then $|FS^*/S^*| \leq 4$ and $C_{S^*}(y) \geq F \cap S^*$ indicates that $|C_{S^*}(y)| \geq 2^{3a/2-2}$. On the other hand, the 2-rank of $C_{S^*}(y)$ is at most a . Hence, as a is even, $a = 4$, $|FS^*/S^*| = 4$ and $|F| = 16$. But then $C_{E_1}(x) \leq F$ or $C_{E_2}(x) \leq F$. As F contains a graph automorphism this is impossible. Thus every elementary abelian subgroup of Q of order at least $2^{3a/2}$ is contained in S^* . Hence $E_1 \cap Q$ and $E_2 \cap Q$ are the unique elementary abelian subgroups of Q of their order and so $U = (E_1 \cap Q) \cap (E_2 \cap Q)$ is normalized by $N_G(Q)$ which is a contradiction. Therefore we may suppose that Q contains no elements which act as field automorphisms on H^* . In particular, we have $|QS^*/S^*| = 2$. If Q contains a graph automorphism, then Q centralizes U and therefore $N_{H^*}(U)$ normalizes Q . But then we have $[N_H^*(Q), Q] \leq Q \cap H^* \leq S^*$ contrary to the fact that the graph automorphism does not centralize a torus of H^* when $a > 2$. Thus the elements of $Q \setminus S^*$ induce graph-field automorphisms on H^* . Now U is a normal subgroup of Q and that Q has no other normal elementary abelian subgroup of the same order. Thus U is normal in $N_G(Q)$ in this case as well contrary to our assumption. Consequently $a = 2$.

Now suppose that $a = 2$. Then we have $|Q/S^*| \leq 4$ and by (3.13.1) $Q \geq U$ with U elementary abelian of order 4. Let $W = \langle U^{N_G(Q)} \rangle$ and assume that $W \neq U$. Suppose that $W \leq S^*$. Then W centralizes U and is therefore abelian and hence elementary abelian. So without loss of generality we may assume that $W \leq E_1$. Then H is contained in H^* extended by a field automorphism. If $W = E_1$, then $N_G(Q) \leq N_G(E_1) \leq H$, which is impossible. Hence W has order 2^3 . Note that W is normal in S and so $N_G(W)$ is of characteristic 2. As $C_S(W) = E_1$, we get that $N_G(W) = N_{N_G(W)}(E_1)C_G(W)$. As $|E_1 : W| = 2$, we have that $C_G(W)$ has a normal 2-complement J . But then J is also normal in $N_G(W)$, which is of characteristic 2. So $J = 1$ and $C_G(W) = E_1$. This now shows that $N_G(Q) \leq N_G(W) \leq N_G(E_1) \leq H$, a contradiction. Thus we have $W \not\leq S^*$. Then $W \cap S^*$ contains elements of order 4. Since W is generated by involutions it follows that W is not abelian and U is not in the centre of W . Let $u \in W$ be conjugate to an element of U and assume that $u \notin H^*$ and u does not centralize U . Then $C_{H^*}(u) \cong \mathrm{PSU}_3(2)$ or $\mathrm{PSL}_3(2)$. As u is conjugate to an element of $Z(Q)$, W is normalized by such a group. Since $\mathrm{PSU}_3(2)$ has no GF(2)-representations of dimension less than 8 and $|S| \leq 2^8$ is non-abelian, we must have that u acts as the field automorphism of H^* . Because $|N_G(Q)/Q| \geq |\mathrm{PSL}_3(2)|_2 = 2^3$, we have that $|Q| \leq 2^5$. This means that $Q = W$ has order 2^5 . Let $L \cong \mathrm{PSL}_3(2)$ act on Q . Then $[L, Z(Q)] = 1$. If $|\Omega_1(Z(Q))| = 4$, then L acts transitively on $Q/\Omega_1(Z(Q))$ and so as Q is generated by involutions Q is elementary abelian, a contradiction. So $Z(Q)$ is cyclic. If X is normal in Q and L -invariant and $|X| = 4$, then L acts transitively on Q/X and so $X \leq Z(Q)$. Hence in any case there is $Y \leq Q$, $Z(Q) \leq Y$, $|Y/Z(Q)| = 8$ and L acts transitively on $Y/Z(Q)$. As L acts 2-transitively on $Y/Z(Q)$, we then get that Y is abelian. As Q is non-abelian, $Y \neq Q$ and so Y is elementary abelian. But now $[Q, Y]$ is trivial or isomorphic to $Y/Z(Q)$. The latter is not possible and so $[Y, Q] = 1$, which gives the contradiction Q abelian. Therefore (3.13.2) holds when $Q \not\leq H^*$.

Assume now that $Q \leq H^*$. Then $[U, Q] = 1$ and so $U \leq Z(Q)$. If $\Omega_1(Z(Q)) = U$ we are done, so $\Omega_1(Z(Q)) \neq U$. Since Q is normalized by $N_G(S^*)$, we now may suppose that $Q = E_1$. But then by assumption $N_G(Q) \leq H$, which is a contradiction. This proves (3.13.2). ■

We now consider $N_G(U)$. As U is normal in S and G has parabolic characteristic 2, $C_G(O_2(N_G(U))) \leq O_2(N_G(U))$.

Suppose that $O_2(N_G(U)) \leq S^*$. Then, by Lemma 3.12, $S^* = O_2(N_G(U))$ and we conclude that $N_G(U)$ has a subgroup N of index 2 which normalizes E_1 . But then $N \leq H$ by hypothesis and so also

$$N_G(Q) \leq N_G(U) = NS \leq H$$

by (3.13.2), a contradiction.

Thus $O_2(N_G(U)) \not\leq S^*$. Then, by Lemma 3.12, we have $a = 2$. Let T be a complement to S^* in $N_{H^*}(S^*)$. Then T normalizes U and hence it also normalizes $O_2(N_G(U))$. But then

$$[T, O_2(N_G(U))] \leq O_2(N_G(U)) \cap H^* \leq S^*$$

and $O_2(N_G(U))$ contains an element which acts as a graph automorphism on H^* . Now just as in the last paragraph $E_1 \not\leq O_2(N_G(U))$ and so $U = \Omega_1(N_G(U) \cap S^*)$. As $[E_1, O_2(N_G(U))]E_1 = S^*$, we have $O_2(N_G(U)) \cap S^* \cong 4 \times 4$ is characteristic in $O_2(N_G(U))$. But then $|N_G(U)/C_G(U)|_{2'} = 3$ and so $N_G(U) = C_G(U)TS = C_G(U)N_H(U)$. As $C_G(U)$ is a 2-group we get $N_G(U) = TS \leq H$. Using (3.13.2) we get $N_G(Q) \leq N_G(U) \leq H$. \square

Lemma 3.14. *If $H^* \cong \mathrm{PSL}_3(2^a)$, then $a = 2$.*

Proof. By Lemma 3.11 we have $a > 1$. So assume that $a > 2$. By Lemma 3.13 there is an elementary abelian subgroup $E_1 \leq S^*$ of order 2^{2a} such that $N_G(E_1) \not\leq H$ and $O^{2'}(N_H(E_1)/E_1) \cong \mathrm{SL}_2(2^a)$. Let $i \in N_S(E_1)$ be an involution which projects non-trivially in $O^{2'}(N_H(E_1)/E_1)$. Then $C_{S \cap N_H(E_1)}(i)$ contains an elementary abelian subgroup E_2 and $C_{S \cap N_H(E_1)}(i)/E_2E_1$ is cyclic. Suppose that $j \in N_G(E_1)$ is an involution which projects into $N_S(E_1)/E_1$, but not into $O^{2'}(N_H(E_1)/E_1)$. Then j acts as a field automorphism on $\mathrm{PSL}_3(2^a)$ and so $C_{N_S(E_1)}(j)$ is an extension of Sylow 2-subgroup of $\mathrm{PSL}_3(2^{\frac{a}{2}})$ by a cyclic group. Since in this case $a \geq 4$, this latter group does not contain an elementary abelian group with cyclic factor group. Hence i and j are not conjugate in $N_G(E_1)$. Using Thompson's Transfer Lemma [8, Theorem 15.16], there is a normal subgroup N of $N_G(E_1)$ which has Sylow 2-subgroup contained in $N_{H^*}(E_1)/E_1$. Thus, employing Lemma 2.2, $O^{2'}(N_{H^*}(E_1))C_G(E_1)$ is a normal subgroup of $N_G(E_1)$.

Suppose that $1 \neq \omega \in C_G(E_1)$ has odd order. Then $[Z(S^*), \omega] = 1$. As $a \geq 3$, we get with Lemma 3.12 that $O_2(C_G(Z(S^*))) = S^*$. Thus ω normalizes S^* and so Lemma 2.3 and $F^*(N_G(Z(S^*))) = O_2(N_G(Z(S^*)))$ together imply $\omega = 1$. Therefore $O^{2'}(N_{H^*}(E_1))$ is a normal subgroup of $N_G(E_1)$ and by the Frattini Argument

$$N_G(E_1) = N_G(S^*)O^{2'}(N_{H^*}(E_1)).$$

Similarly we have

$$N_G(E_2) = N_G(S^*)O^{2'}(N_{H^*}(E_2)).$$

In particular, we now have $N_G(S^*)$ normalizes

$$\langle O^{2'}(N_{H^*}(E_1)), O^{2'}(N_{H^*}(E_2)) \rangle = H^*.$$

But then $N_G(E_1)$ and $N_G(E_2)$ are contained in H and this contradicts Lemma 3.13. \square

Proposition 3.15. *If $H^* \cong \mathrm{PSL}_3(2^a)$, then $a = 2$ and $G \cong \mathrm{Mat}(23)$.*

Proof. By Lemma 3.14 we have $H^* \cong \mathrm{PSL}_3(4)$ and because of Lemma 3.13 we may assume that $N_G(E_1) \not\leq H$. Suppose that $i \in \{1, 2\}$ and $1 \neq \omega \in C_G(E_i)$ is of odd order. Then $[U, \omega] = 1$. As $Z(Q) \cap U \neq 1$, we have that $C_G(E_i) \leq N_G(Q)$. In particular ω normalizes $Q \cap N_G(E_i)$ and so centralizes $Q \cap N_G(E_i)$. As $|Q : Q \cap N_G(E_i)| \leq 2$, we get $[Q, \omega] = 1$, a contradiction. Therefore $C_G(E_i) = E_i$ for both $i = 1, 2$. This yields that $N_G(E_i)/E_i$ is isomorphic to a subgroup of $\mathrm{SL}_4(2)$ which strictly contains $\mathrm{Alt}(5)$ or $\mathrm{Sym}(5)$ as a subgroup of odd index. Inspection of the subgroups of $\mathrm{SL}_4(2)$ shows that

$$N_G(E_1)/E_1 \cong \mathrm{Alt}(5) \times 3, (\mathrm{Alt}(5) \times 3) : 2 \text{ or } \mathrm{Alt}(7).$$

If $N_{H^*}(E_i)$ is normal in $N_G(E_i)$ for both $i = 1, 2$, then $N_G(E_i) = N_{N_G(E_i)}(S^*)N_{H^*}(E_i)$. This then gives $N_{N_G(E_1)}(S^*) = N_{N_G(E_2)}(S^*)$ which then means that $N_{N_G(E_1)}(S^*)$ normalizes $\langle N_{H^*}(E_1), N_{H^*}(E_2) \rangle = H^*$ and so $N_G(E_i) \leq N_G(H^*) = H$ which is a contradiction. Hence we may assume that

$$N_G(E_1)/E_1 \cong \mathrm{Alt}(7).$$

Now taking $z \in Z(S)$, we get that $C_{N_G(E_1)}(z) = E_1 L$ where $L \cong \mathrm{SL}_3(2)$ and so $Q = E_1$ and $C_G(z) = E_1 L$. Hence $C_G(z) \leq N_G(E_1)$. It also follows that $N_G(E_2)/E_2 \not\cong \mathrm{Alt}(7)$ (for otherwise $E_1 = Q = E_2$) and so $N_G(E_2)$ normalizes $N_{H^*}(E_2)$. Since $N_{N_G(E_1)}(E_2) = N_{N_G(E_1)}(E_1 \cap E_2)$ and $\mathrm{Alt}(7)$ acts transitively on the subgroups of E_1 of order 2^2 , we have

$$N_{N_G(E_1)}(E_2)/E_1 \approx (3 \times \mathrm{Alt}(4)) : 2$$

and $N_{N_G(E_1)}(E_2) = N_G(E_1) \cap N_G(E_2)$. It follows that

$$N_G(E_2)/E_2 \approx (3 \times \mathrm{Alt}(5)) : 2.$$

Since $N_G(E_1) = \langle N_{N_G(E_1)}(E_2), N_{H^*}(E_1) \rangle$, we have that $H \approx \mathrm{PSL}_3(4) : 2$.

Let $B \leq N_G(E_1)$ be such that $B/E_1 \cong \mathrm{Alt}(6)$ and $B \cap H^* \approx 2^4 : \mathrm{SL}_2(4)$. Let $U = \langle H^*, B \rangle$. We have $N_B(E_2) = N_B(E_1 \cap E_2) \approx 2^4 : \mathrm{Sym}(4)$ and $N_B(E_2) \not\leq H$. We set $C = N_B(E_2)N_{H^*}(E_2) \leq N_G(E_2)$. Then $C/E_2 \cong \mathrm{Sym}(5)$. Applying Lemma 2.11 gives $U \cong \mathrm{Mat}(22)$. Now we consider the triangle of groups consisting of U , $N_G(E_1)$ and

$N_G(E_2)$ and apply Lemma 2.12 with $B = N_G(E_1)$ and $P = U$ to get $M = \langle P, U \rangle \cong \text{Mat}(23)$. In particular, we now know that G has one conjugacy class of involutions, the fusion of these involutions is controlled in M and $C_G(x) \leq M$ for all involutions x in M . Thus, if $M < G$, then M is strongly 2-embedded in G ([8, Proposition 17.11]). Since, by [3], G does not have a strongly 2-embedded subgroup we infer that $G = M$ and this completes the proof of the proposition. \square

We conclude this section by proving Theorem 1.1 when $p = 2$.

Proof of Theorem 1.1 when $p = 2$. Lemma 2.4(i) and (ii) provides a list of candidates for H^* . The configurations with $H^* \cong \Omega_6^+(2)$, $\text{PSU}_4(2)$, $\text{PSU}_5(2)$, $\text{Sp}_6(2)$, ${}^2\text{F}_4(2)$, $\text{Sp}_4(2^a)'$ for some $a \geq 2$ and $\text{PSL}_3(2^a)$ for $a \neq 2$ are shown to be impossible in Lemmas 3.2, 3.4, 3.5, 3.7, 3.8, 3.10 and 3.14 respectively. The remaining possibilities are that $H^* \cong \Omega_8^+(2)$, $\text{G}_2(2)'$, $\text{Sp}_4(2)'$ or $\text{PSL}_3(4)$. In these cases Propositions 3.3, 3.6, 3.9 and 3.15 show that $F^*(G)$ is as described in Theorem 1.1 (ii). \square

4. THE CONFIGURATIONS WITH $p = 3$

In this section we assume that G and H are as in the statement of the main theorem and that in addition $p = 3$. We adopt the notation from Section 3 and investigate each of the groups listed in Lemma 2.4 with the exception of $\text{PSL}_3(3^a)$.

Lemma 4.1. *We have $H^* \not\cong \text{PSp}_4(3)$.*

Proof. Suppose that $H^* \cong \text{PSp}_4(3)$. Then $S = S^* \leq H^*$ and so, letting $Z = Z(S)$, we have $Q \leq O_3(N_H(Z))$. Since $O_3(N_H(Z)) \approx 3_+^{1+2}$ and $N_{H^*}(Z)/O_3(N_H(Z)) \cong \text{SL}_2(3)$, we have $Q = O_3(N_H(Z))$. Therefore $\text{Out}(Q) \cong \text{GL}_2(3)$ and so $N_H(Z)$ is normalized by $N_G(Q)$. Then $N_G(Q) = N_{N_G(Q)}(S)N_H(Q)$ and $N_{N_G(Q)}(S)$ normalizes the unique abelian subgroup E of S of order 3^3 . From the structure of $\text{PSp}_4(3)$, we get $N_{H^*}(E)/E \cong \text{Alt}(4)$ and $C_G(E) = E$ as E is normal in S and G has parabolic characteristic 3. Thus $\langle N_{H^*}(E), N_{N_G(Q)}(S) \rangle$ embeds into $\text{GL}_3(3)$ and has Sylow 3-subgroups of order 3 and non-trivial Sylow 2-subgroups. Surveying the maximal subgroups of $\text{GL}_3(3)$ [4] shows that $N_{N_G(Q)}(S)$ normalizes $N_{H^*}(E)$. But then $N_{N_G(Q)}(S)$ normalizes $\langle N_{H^*}(E), N_{H^*}(Z) \rangle = H^*$. Therefore $N_G(Q) \leq H$, which is a contradiction. \square

Lemma 4.2. *We have $H^* \not\cong \text{G}_2(3^a)$.*

Proof. Suppose that $H^* \cong \text{G}_2(3^a)$. Then $Z(S) \cap S^*$ contains both long and short root elements z_1 and z_2 in $Z(S^*)$. Thus $N_G(Q)$ contains $\langle C_{H^*}(z_1), C_{H^*}(z_2) \rangle = H^*$ which is impossible. \square

Proposition 4.3. Suppose that $H^* \cong \mathrm{PSL}_4(3)$ or $\mathrm{PSU}_4(3)$. Then $H^* \cong \mathrm{PSU}_6(2)$, $\mathrm{F}_4(2)$, McL or Co_2 .

Proof. Suppose that $H^* \cong \mathrm{PSL}_4(3)$ or $\mathrm{PSU}_4(3)$. Then $S = S^*$ and we have $z \in Q \cap Z(S)$ with $O_3(C_H(z)) \cong 3_+^{1+4}$. Furthermore $E = J(S)$ is elementary abelian of order 3^4 and (see [4, pages 69 and 52])

$$N_{H^*}(E)/E \cong \begin{cases} (\mathrm{SL}_2(3) * \mathrm{SL}_2(3)) : 2 & \text{if } H^* \cong \mathrm{PSL}_4(3) \\ \mathrm{PSL}(2, 9) \cong \mathrm{Alt}(6) & \text{if } H^* \cong \mathrm{PSU}_4(3) \end{cases}.$$

In both cases an inspection of the maximal subgroups of $\mathrm{GL}_4(3)$ [4] yields

(4.3.1) $O^{3'}(N_H(E))$ is normal in $N_G(E)$ and $N_G(E) = N_G(S)O^{3'}(N_H(E))$.

We next determine Q . Suppose that $Q \neq O_3(C_H(z))$. Then Q is a normal subgroup of $C_H(z)$, which is properly contained in $O_3(C_H(z))$ and different from $\langle z \rangle$. Then Lemma 2.9 gives $H^* \cong \mathrm{PSL}_4(3)$ and $|Q| = 3^3$. Furthermore, by Lemma 2.10, there are exactly 4 possibilities for Q , two of them are extraspecial and two are elementary abelian. If $Q \cong 3_+^{1+2}$, then $C_{O_3(C_H(z))}(Q)Q = O_3(C_H(z)) > Q$, which is a contradiction. Hence Q is elementary abelian and $C_G(Q) = Q$. Furthermore, $N_H(Q)/Q \cong \mathrm{SL}_3(3)$ and so, as $N_G(Q) > N_H(Q)$ we must have $N_G(Q)/N_H(Q) = 2$ and $Z(N_G(Q)/Q) = 2$. Let Y be the preimage of $Z(N_G(Q)/Q)$. Then Y normalizes $O_3(C_H(z))$ and hence Y normalizes $N_H(E_2)$ where E_2 is an elementary abelian normal subgroup of $N_G(\langle z \rangle)$ contained in Q with $E_2 \neq Q$. But then Y normalizes $H^* = \langle N_{H^*}(Q), N_{H^*}(E_2) \rangle$ and consequently $N_G(Q) \leq H$, which is a contradiction.

Therefore $Q = O_3(C_H(z))$ is extraspecial of order 3^5 and exponent 3. This yields that $N_G(Q)/Q$ is isomorphic to a subgroup of $\mathrm{GSp}_4(3)$, which has a Sylow 3-subgroup of order 3. Employing either [4] or [12] we get that one of the following holds:

- (1) $N_G(Q)/Q \cong 2_-^{1+4}.\mathrm{Alt}(5)$ or $2_-^{1+4}.\mathrm{Sym}(5)$;
- (2) $E(N_G(Q)/Q) \cong \mathrm{SL}_2(5)$; or
- (3) $|N_G(Q)/Q| = 2^a \cdot 3$.

If (1) occurs, then, as $Z(S)$ is not weakly closed in S with respect to G , [14] yields $G \cong \mathrm{Co}_2$.

Suppose we have possibility (2). Assume further that $H^* \cong \mathrm{PSL}_4(3)$. We will show $N_G(S) \leq H$.

We know that $N_G(S)$ normalizes E and by (4.3.1) also $O^{3'}(N_H(E))$. In S there are two elementary abelian subgroups E_1, E_2 of order 3^3

such that, for $i = 1, 2$, $U_i = \langle Q^g \mid Z(Q)^g \leq E_i \rangle$ satisfies

$$U_i/E_i \cong \mathrm{SL}_3(3).$$

In fact, E_1 is the group of transvections to a point and E_2 the group of transvections to a hyperplane containing this point. Hence E_iE/E correspond to the two subgroups of order three in S/E , which act quadratically on E . In particular $N_G(S)$ acts on $\{E_1E, E_2E\}$. We have that $O^{3'}(N_H(E))$ contains an involution x which inverts E . Let $M = N_{N_G(S)}(E_1E)$. We factor $M = C_M(x)E$. Then, for $i = 1, 2$, M normalizes $Z(E_iE) = E_i \cap E$ which has order 3^2 . Now $E_i = C_{E_iE}(x)(E_i \cap E)$ is normalized by $C_M(x)$. Since E normalizes E_i , we infer that E_i is normalized by M . Therefore $N_G(S)$ permutes $\{E_1, E_2\}$ and normalizes $\langle U_1, U_2 \rangle = H^*$. Hence by assumption we then have that $N_G(S) \leq H = N_G(H^*)$.

Now generally if (2) holds, then

$$N_G(Q) = \langle N_H(Q), N_{N_G(Q)}(S) \rangle,$$

as $N_H(Q)/Q \cap E(N_G(Q)/Q) \cong \mathrm{SL}_2(3)$ and $N_{E(N_G(Q)/Q)}(S/Q) \approx 3 : 4$ and together these groups generate $E(N_G(Q)/Q)$. Hence as $N_G(Q) \not\leq H$, we get that $F^*(H) \cong \mathrm{PSU}_4(3)$ and so $E(N_G(E)/E) \cong \mathrm{Alt}(6)$. Finally, using Lemma 2.15 yields $F^*(G) \cong \mathrm{McL}$.

So we may assume that we have possibility (3). The Frattini Argument delivers

$$N_G(O^{3'}(N_H(Q))) = N_{N_G(O^{3'}(N_H(Q)))}(S)O^{3'}(N_H(Q)).$$

The left factor normalizes $O^{3'}(N_H(E))$ by (4.3.1). Therefore $N_G(O^{3'}(N_H(Q)))$ normalizes $\langle O^{3'}(N_H(E)), O^{3'}(N_H(Q)) \rangle = H^*$ and so is contained in H . Thus $N_{N_G(Q)}(O^{3'}(N_H(Q))) \leq H$. Using this information and when inspecting the subgroups of $\mathrm{GSp}_4(3)$ given in [4] we obtain

$$O^{3'}(N_G(Q)/Q)N_H(S) \leq R \cong (Q_8 \times Q_8).\mathrm{Sym}(3).$$

Hence, as $N_G(Q) \not\leq H$, we get that R is isomorphic to a subgroup of $N_G(Q)/Q$. In particular $N_G(Q)/Q$ is a subgroup of the subgroup of $\mathrm{GSp}_4(3)$ which preserves a decomposition of the natural 4-dimensional symplectic space over $\mathrm{GF}(3)$ into a perpendicular sum of two non-degenerate 2-spaces. We further see that $O^{3'}(N_G(Q)/Q)$ is isomorphic to a subgroup of $\mathrm{Sp}_2(3) \times \mathrm{Sp}_2(3)$ which projects nontrivially on to both direct factors. In particular $O^{3'}(N_G(Q)/Q)$ contains a normal subgroup isomorphic to $Q_8 \times Q_8$.

Therefore, if either $H^* \cong \mathrm{PSU}_4(3)$ or $H^* \cong \mathrm{PSL}_4(3)$, then $Z(Q)$ is not weakly closed in S . Hence in case (3) we have that $F^*(G) \cong \mathrm{PSU}_6(2)$ or $\mathrm{F}_4(2)$ by Lemma 2.13. \square

Proposition 4.4. Suppose that $H^* \cong \Omega_7(3)$. Then $G^* \cong {}^2\mathrm{E}_6(2)$ or $\mathrm{M}(22)$.

Proof. Again $S = S^*$. We set $Z = Z(S)$ and note that

$$N_{H^*}(Z) \approx 3^{1+6}_+ \cdot (\mathrm{SL}_2(3) \times \Omega_3(3)) \cdot 2.$$

As a module for this group $O_3(C_H(Z))/Z$ is the tensor product of the natural $\mathrm{SL}_2(3)$ -module with the 3-dimensional orthogonal $\Omega_3(3)$ -module and this is an irreducible action. Therefore $Q = O_3(N_{H^*}(Z))$. Inspection of the irreducible subgroups of $\mathrm{Sp}_6(3)$ (see [4]) shows that $N_G(Q)/Q$ is a subgroup of $U = (\mathrm{Sp}_2(3) \wr \mathrm{Sym}(3)) : 2$. As obviously $\Omega_1(O_2(U)) \leq N_{H^*}(Q)/Q$ we either see that the assumptions of Lemma 2.16 or Lemma 2.17 are satisfied and so we have the assertion or $|N_G(Q) : N_{H^*}(Q)| = 2$. So assume that $|N_G(Q) : N_{H^*}(Q)| = 2$. Then $N_G(Q)/Q \cong \mathrm{GL}_2(3) \times \mathrm{Sym}(4)$ and $N_G(Q) = N_G(S)N_{H^*}(Q)$. Let $P \leq N_G(Q)$ be such that $P/O_3(P) \cong \mathrm{GL}_2(3) \times 2$ and note that $O_3(P) = QO_3(L)$ where L is the parabolic subgroup of H^* which contains S and has shape $3^{3+3} : \mathrm{SL}_3(3)$. We have that the preimage of $C_{Q/Z(Q)}(O_3(P))$ is equal to $E = Z(O_3(L))$ is elementary abelian of order 3^3 and is normalized by $N_G(S)$. As E is normal in S , $O_3(N_G(E)) = O_3(N_{H^*}(E))$. This yields that $N_{H^*}(E)$ is normal in $N_G(E)$, as $N_{H^*}(E) = O^{3'}(N_G(E))$. Then $N_G(S)$ normalizes $\langle N_{H^*}(E), N_{H^*}(Z(Q)) \rangle = H^*$. But then by assumption $N_G(Q) = N_G(S)N_{H^*}(Q) \leq H$, which is a contradiction. This proves the proposition. \square

We finally consider the configurations with $H^* \cong \mathrm{P}\Omega_8^+(3)$ and do this though a series of lemmas. Set $Z = Z(S^*) = Z(S)$. We have that Z has order 3 and

$$N_{H^*}(Z)/O_3(N_{H^*}(Z)) \approx (\mathrm{SL}_2(3) * \mathrm{SL}_2(3) * \mathrm{SL}_2(3)) : 2 \approx 2^{1+6}_-.3^3.2,$$

as can be seen in [4]. We also recall that H/H^* embeds into $\mathrm{Out}(H) \cong \mathrm{Sym}(4)$. The action of $N_{H^*}(Z)$ on $O_3(N_{H^*}(Z))$ is as a tensor product of the natural $\mathrm{SL}_2(3)$ -module with the four-dimensional orthogonal module for $\mathrm{O}_4^+(3)$. In particular, $N_{H^*}(Z)$ acts irreducibly on $O_3(N_{H^*}(Z))/Z$ which has order 3^8 .

Hence we get

Lemma 4.5. Suppose that $H^* \cong \mathrm{P}\Omega_8^+(3)$. Then

- (i) $Q = O_3(N_{H^*}(Z))$ is extraspecial of order 3^9 of exponent 3 and $N_G(Q)/Q$ is isomorphic to a subgroup of $\mathrm{GSp}_8(3)$.
- (ii) S^*/Q is elementary abelian of order 3^3 and $|S/S^*| \leq 3$.
- (iii) $N_{H^*}(S^*)/S^*$ has order 2.

- (iv) $N_H(Q)/Q$ is isomorphic to a subgroup of a group of shape $(\mathrm{GL}_2(3) * \mathrm{GL}_2(3) * \mathrm{GL}_2(3)).\mathrm{Sym}(3)$ embedded in $\mathrm{GSp}_8(3)$. In particular, $N_H(Q)$ is a $\{2, 3\}$ -group.

□

We remark that the subgroup of $\mathrm{GSp}_8(3)$ in Lemma 4.5 (iv) is precisely (and uniquely) described in [19, Section 3].

Lemma 4.6. *Suppose that $H^* \cong \mathrm{P}\Omega_8^+(3)$ and $E \leq S^*$ is an elementary abelian subgroup of S^* of order 3^6 such that $N_{H^*}(E)/E \cong \Omega_6^+(3)$. Then $N_{H^*}(E)$ is normal in $N_G(E)$ and $N_G(E) = N_{N_G(E)}(S^*)N_{H^*}(E)$.*

Proof. First suppose that ω is a $3'$ -element which centralizes E . Then $[\omega, Q \cap E] = [\omega, Z] = 1$ and so ω normalizes Q and centralizes a maximal abelian subgroup in Q . Thus $[Q, \omega] = 1$ by Lemma 2.3 and consequently $\omega = 1$.

Let $e \in E$ correspond to a non-singular point in E and assume that e is conjugate to z in $N_G(E)$. Then $C_{N_{H^*}(E)}(e)/E$ has a normal subgroup isomorphic to $\Omega_5(3) \cong \mathrm{PSp}_4(3)$. As $|S : Q| \leq 3^4$, we see that $E \leq O_3(C_G(e))$. But Q does not contain an elementary abelian group of order 3^6 . So $N_{H^*}(E)$ controls fusion of the $N_G(E)$ -conjugates of z in E and this yields $N_{H^*}(E) = \langle Q^{N_G(E)} \rangle$. In particular $N_{H^*}(E)$ is normal in $N_G(E)$ and $N_G(E) = N_{H^*}(E)N_{N_G(E)}(S^*)$ as claimed. □

Lemma 4.7. *Suppose that $H^* \cong \mathrm{P}\Omega_8^+(3)$. Then $N_G(S^*) = N_H(S^*)$.*

Proof. We have that Q is normalized by $N_G(S^*)$ and S^*/Q is elementary abelian of order 3^3 . In S^* there are three elementary abelian subgroups of order 3^6 , E_1, E_2, E_3 , whose normalizer in H^* involves $\Omega_6^+(3)$. Furthermore the groups QE_i , $i = 1, 2, 3$, correspond to three different subgroups of order three in S^*/Q . By [19, Lemma 3.1(i)] these are the only three subgroups of order three in S^*/Q , which centralize an elementary abelian subgroup of order 3^5 in Q . Therefore $N_G(S^*)$ permutes E_1, E_2 and E_3 . Hence, by Lemma 4.6, $N_G(S^*)$ normalizes $\langle N_{H^*}(E_i) \mid i = 1, 2, 3 \rangle = H^*$. By our general assumption we then have $N_G(S^*) \leq H$. □

Lemma 4.8. *Suppose that $H^* \cong \mathrm{P}\Omega_8^+(3)$. Then $N_G(Z_2(S)) \leq H$.*

Proof. From [19, Lemma 3.1 (v)] $Z_2(S)$ has order 9. Assume that $g \in H^*$ and $z^g \in Z_2(S)$. Then $z^g = z^h$ for some $h \in H^*$ and therefore

$$P = \langle Q^g \mid g \in G, z^g \in Z_2(S) \rangle \leq H^*$$

which means that P is normal in $N_G(Z_2(S))$. Hence, from the structure of $\mathrm{P}\Omega_8^+(3)$ we have $P \geq S^*$ and $N_G(Z_2(S)) = PN_G(S^*)$. Finally Lemma 4.7 yields then $N_G(Z_2(S)) \leq H$. □

Set

$$X = O_{3,2}(N_H(Q)).$$

Our objective over the next few lemmas is to show that $N_G(Q) = N_G(X)$.

Lemma 4.9. *Suppose that $H^* \cong P\Omega_8^+(3)$. Then X/Q is extraspecial of order 2^7 and of $--$ -type and one of the following holds:*

- (i) $N_{C_G(Z(Q))}(X)/X \cong \text{PSU}_4(2)$ or $3_+^{1+2}.\text{SL}_2(3)$; or
- (ii) $N_{C_G(Z(Q))}(X) \leq H$.

Proof. We have already commented that X/Q is a central product of three subgroups isomorphic to Q_8 . By Lemma 2.5, $N_{C_G(Z(Q))}(X)/X$ is isomorphic to a subgroup of $\text{PSU}_4(2)$ which has order divisible by 3^3 . If $N_{C_G(Z(Q))}X/X$ normalizes S^*X/X , then

$$N_{C_G(Z(Q))}(X) \leq N_G(S^*)X \leq H$$

by Lemma 4.8. This is (ii). Employing the subgroup structure of $\text{PSU}_4(2)$ as given in [4] now delivers the assertion. \square

Lemma 4.10. *Suppose that $H^* \cong P\Omega_8^+(3)$. If $N_{C_G(Z(Q))}(X)/X \cong \text{PSU}_4(2)$, then $N_G(X) = N_G(Q)$.*

Proof. Set $U = N_{C_G(Z(Q))}(X)$. As $U/X \cong \text{PSU}_4(2)$, we have

$$N_U(XS^*)/X \cong 3^3 : \text{Sym}(4).$$

By Lemma 4.7 $N_U(XS^*) = U \cap H$. Since S acts irreducibly on X/X' , we have that $N_X(Z_2(S)) = X'$. Hence using Lemma 4.8, we see that $Z_2(S)$ has 64×40 conjugates under the conjugation action of U . On the other hand, the number of conjugates is at most $(3^8 - 1)/2 = 40 \times 82$. It follows, again using Lemma 4.8, that $Z_2(S)^U = Z_2(S)^{C_G(Z(Q))}$ and $U = C_G(Z(Q))$. \square

Lemma 4.11. *Suppose that $H^* \cong P\Omega_8^+(3)$ and Y is a $N_G(Q)$ -conjugate of X . Let $i \in Y$ be an involution with $iQ \notin Z(Y/Q)$ and P be the preimage of $C_{N_G(Q)/X'}(\langle i \rangle, X')$. Then $P \leq N_G(Y)$.*

Proof. As X' inverts $Q/Z(Q)$, we have that X' is normal in $N_G(Q)$. To reach the conclusion of the lemma we may as well suppose that $Y = X$. We set $\overline{C_G(Z(Q))} = C_G(Z(Q))/Q$ and identify it with a subgroup of $\text{Sp}_8(3)$. Let \bar{i} be an involution in \overline{X} . Since $Z(\overline{X}) = \langle \bar{j} \rangle$ acts fixed-point-freely on $Q/Z(Q)$ and \bar{i} and $\bar{i}\bar{j}$ are \overline{X} -conjugate, we have $|[Q/Z(Q), \bar{i}]| = 3^4$. This shows that in $\text{Sp}_8(3)$ the group $C_{\overline{C_G(Z(Q))}}(\bar{i})$ is contained in the subgroup $\text{Sp}_4(3) \times \text{Sp}_4(3)$ which preserves the decomposition of the natural $\text{Sp}_8(3)$ -module in to a perpendicular sum of two non-degenerate 4-spaces. The extraspecial group \overline{X} contains 55 involutions and under

the action of $\overline{N_H(Q)}$ we see that $\langle \bar{i}, \bar{j} \rangle$ has either 27 or 9 conjugates, depending on whether 3 divides $|H/H^*|$ or not. Hence $|C_{\overline{N_H(Q)}}(\bar{i})| = 2^b \cdot 3$ where $0 \leq b \leq 4$.

Choose $E \in \{E_1, E_2, E_3\}$ to be an elementary abelian subgroup of S^* of order 3^6 as in Lemma 4.6. Then E normalizes X and, as $\overline{EX} \cong \mathrm{SL}_2(3) * 2_+^{1+4}$ corresponds to an end node of the Dynkin diagram, we have $[\overline{X}, \overline{E}] \cong Q_8$. Furthermore, $C_{\overline{X}}(\overline{E}\bar{E}_j) \cong Q_8$. Thus by counting we see that every non-central involution of \overline{X} is centralized by some $E \in \{E_1, E_2, E_3\}$. In particular, we may assume that E is chosen so that $[i, E] \leq Q$. Then by Lemma 4.6 again we get that \overline{QE} is a Sylow 3-subgroup of $C_{\overline{N_G(Q)}}(\bar{i})$ and $\overline{N_G(QE)} \leq \overline{N_G(Q)} \cap \overline{N_G(E)}$ as $E = C_{QE}(E \cap Q)$ is the unique elementary abelian subgroup of order 3^6 in QE . Hence by Lemma 4.6 and Lemma 4.7 we have $\overline{N_G(QE)} \leq \overline{N_H(E)}$.

Let $k \in \overline{X}$ with $\bar{i}^k = \bar{j}$ and write

$$W = N_{\mathrm{Sp}_8(3)}(\langle \bar{i}, \bar{j} \rangle) = (L_1 \times L_2)\langle k \rangle.$$

where $L_1 \cong \mathrm{Sp}_4(3)$, $L_1^k = L_2$ and $\bar{i} \in L_1$.

Since $\overline{X} \leq W$ and \overline{X} does not centralize \bar{i} , we now see that

$$N_{W'}(\overline{X}) \approx (2_-^{1+4} \times 2_-^{1+4}).\mathrm{Alt}(5)$$

and $\overline{N_H(Q)} \cap N_{W'}(\overline{X})\overline{X} \geq \overline{XE}$.

Suppose that 5 divides the order of $U = C_{\overline{C_G(Q)}}(\bar{i})$. And assume that $U \not\leq \overline{N_H(Q)}$. Then the structure of $\mathrm{Sp}_4(3) \times \mathrm{Sp}_4(3)$ and the fact that 9 does not divide the order of U now gives that $U \leq N_W(\overline{X})$ in which case the lemma holds.

So assume that $|U| = 2^a \cdot 3$ for some suitable a . Recall from Lemma 4.10, we have $N_{\overline{C_G(Z(Q))}}(\overline{X})/\overline{X} \not\cong \mathrm{PSU}_4(2)$.

From now on we assume the lemma is false in seek a contradiction. Then $U \not\leq N_W(\overline{X})$. As $U \leq L_1 L_2$, we may project $C_{\overline{X}}(\bar{i})\overline{E} \approx 2_-^{1+4} \cdot 3$ on to the first factor (say) and deduce from the subgroup structure of $\mathrm{Sp}_4(3)$ and the fact that we know $|U| = 2^a \cdot 3$ that U normalizes $C_{\overline{X}}(\bar{i})L_2$. We may therefore assume that

$$1 \neq [U, C_{\overline{X}}(\bar{i})] \cap L_1 \leq O_2(N_{W'}(\overline{X})) \cap L_1.$$

Moreover, as 3 divides $|U|$, $([U, C_{\overline{X}}(\bar{i})] \cap L_1)/\langle \bar{i} \rangle$ is elementary abelian of order 4. This shows that $N_{\overline{C_G(Z(Q))}}(X)/X$ contains an elementary abelian subgroup of order 4, which by Lemma 4.9 implies $N_{\overline{C_G(Z(Q))}}(X) \leq$

H . But there is no $\text{Alt}(4)$ in $N_H(X)/X$ with EQ/Q as a Sylow 3-subgroup and so we have a contradiction. (Recall $N_{H^*}(X)/X \approx 3^3 : 2$.) So $U \leq N_W(\overline{X})$. \square

Lemma 4.12. *Suppose that $H^* \cong P\Omega_8^+(3)$ and Y is $N_G(Q)$ -conjugate to X in $N_G(Q)$. Then Y weakly closed in $N_G(Y)$ with respect to $C_G(Z(Q))$. In particular $|N_G(Q) : N_G(X)|$ is odd.*

Proof. Again it suffices to prove the result for X . Suppose that $X^g \leq N_G(X)$ with $[X, X^g] \leq X \cap X^g$ and $X \neq X^g$ with $g \in N_G(Q) \setminus H$. By Lemma 4.10 we have $N_G(X)/X \not\cong \text{PSU}_4(2)$.

By Lemma 2.5 there are no transvections in $C_G(Z(Q))$ on X/X' . Hence, by Lemma 2.6(i), $[X, X^g]$ contains an involution i with $iQ \notin Z(X/Q)$. Therefore Lemma 4.11 yields that $C_{N_G(Q)}(\langle i, Q \rangle) \leq N_G(X) \cap N_G(X^g)$.

Assume that $N_{C_G(Z(Q))}(X)/X \cong 3_+^{1+2} : \text{SL}_2(3)$. Then $|X^g X/X| = 2$, contradicting the fact that there are no transvections on X/X' .

Therefore Lemma 4.9 implies that $N_G(X) \leq H$ and so $N_{C_G(Z(Q))}(X)/X$ is a subgroup of $3^3 : \text{Sym}(4)$ and $X^g X/X$ is a fours group. Let E be as in Lemma 4.6 be such that $N_{N_G(Q)}(E) \leq N_H(Q)$. Then, as in the previous lemma, we may assume that EQ/Q centralizes iQ . Thus $EQ \leq N_G(X^g) \cap N_G(X)$. From the structure of $N_{H^g}(X^g)/X^g$, we see that $N_{N_G(X^g)}(EQ)/EQ$ contains an elementary abelian group of order 9 and this group is in turn contained in H by Lemma 4.6. Thus $(N_G(X) \cap N_G(X^g))/Q$ contains an elementary abelian group of order 9 and this group normalizes $X^g X$. Since $N_G(X)/X$ is a subgroup of a group of shape $3^3 : \text{Sym}(4)$ an easy calculation shows that it is impossible for a fours group to be normalized by an elementary abelian groups of order 9. This contradiction proves the lemma. \square

Lemma 4.13. *Suppose that $H^* \cong P\Omega_8^+(3)$. Then $N_G(X) = N_G(Q)$.*

Proof. By Lemmas 4.9 and 4.12 we may assume that

$$N_{C_G(Z(Q))}(X)/X \cong 3_+^{1+2} : \text{SL}_2(3)$$

or $N_G(X) = N_H(X)$. Set $\overline{N_G(Q)} = N_G(Q)/X'$.

We first will show that for involutions $i \in X \setminus X'$ we have

(4.13.1) $i^{C_G(Z(Q))} \cap N_G(X) \subseteq X$.

Assume $i^g \in N_G(X) \setminus X$ for some $g \in N_G(Q)$. By Lemmas 4.11 and 4.12 $(X^g \cap X)/Q$ is isomorphic to a subgroup of Q_8 . In particular $|\overline{X} \cap \overline{X^g}| \leq 4$. As $|C_{\overline{X}}(i^g)| = 16$ by Lemma 2.6(i), we see that $N_G(X)/X$ contains a fours group $V = (\overline{X}^g \cap \overline{N_G(X)})\overline{X}/\overline{X}$ and so $N_G(X) =$

$N_H(X)$ and we recall that $N_H(X)/X$ is isomorphic to a subgroup of a groups of shape $3^3.\text{Sym}(4)$. Thus $(N_G(X)/X)/O_3(N_G(X)/X)$ is a subgroup of $\text{Sym}(4)$. In particular, as there is no elementary abelian group of order 8 in $N_G(X)/X$ and, by Lemma 4.12, $N_G(X)$ contains a Sylow 2-subgroup of $N_G(Q)$, we have that $|\overline{X} \cap \overline{X}^g| = 4$ and the preimage of this group in X/Q is quaternion of order 8. Therefore,

$$[C_{\overline{X}}(i^g), \overline{X}^g \cap \overline{N_G(X)}] \leq \overline{X} \cap \overline{X}^g$$

and consists only of elements whose preimages have order 4.

Now the action of $N_G(X)$ on X/Q is uniquely determined. Up to conjugacy, there are exactly two fours groups F_1 and F_2 in $\text{Sym}(4)$ where we assume that F_1 is normal. We have $|C_{\overline{X}}(F_1)| = |C_{\overline{X}}(F_2)| = 8$ from Lemma 2.6(ii). Furthermore, also from Lemma 2.6, the preimage of $C_{\overline{X}}(F_1)$ in X/Q is abelian and so as $X \cap X^g$ is quaternion, we cannot have $V = F_1$. Hence $V = F_2$. On the other hand, by Lemma 2.6 again $[C_{\overline{X}}(i^g), F_2]$ contains elements whose preimages are involutions. Thus we also have $V \neq F_2$. This contradiction shows that all conjugates of i in $N_G(X)$ are contained in X as claimed. ■

Now consider $X^g \cap N_G(X)$ for $g \in N_G(Q)$. By Lemmas 4.11 and 4.12 there are no involutions in $(X \cap X^g) \setminus X'$ and there are no involutions in $(X^g \cap N_G(X)) \setminus X$ by (4.13.1). Thus $(X^g \cap N_G(X))/Q$ is a subgroup of a quaternion group. It follows that $N_G(X^g) \cap X$ has index at least 16 in X for every $g \in N_G(Q) \setminus N_G(X)$. Therefore the number of conjugates of X in $N_G(Q)$ is $1 + k16$ for some integer k . On the other hand, $|N_G(Q) : N_G(X)| |N_G(X) : N_G(Z_2(S^*))| \leq (3^8 - 1)/2$. Hence, as $|N_G(X) : N_G(Z_2(S^*))| = 64$, $|N_G(Q) : N_G(X)| < 52$. Thus the number of conjugates of X in $N_G(Q)$ is 1, 17, 33 or 49. The only one of these numbers which divides $|\text{Sp}_8(3)|$ is 1. Hence X is normal in $N_G(Q)$. □

Proposition 4.14. *Suppose that $H^* \cong \text{P}\Omega_8^+(3)$. Then $F^*(G) \cong \text{F}_2$ or $\text{M}(23)$.*

Proof. By Lemma 4.13, we have that X is normal in $N_G(Q)$. As $N_G(Q) \not\leq H$ and $N_H(Q)$ contains an element which inverts $Z(Q)$, Lemma 4.9 indicates that $N_G(Q)/Q$ is an extension of X by $3_+^{1+2}.\text{GU}_3(2)$ or $\text{PSU}_4(2) : 2$. Now an application of Lemma 2.18 and Lemma 2.19 yield the assertion. □

We now prove Theorem 1.1.

Proof of Theorem 1.1. We have already proved the theorem when $p = 2$ in Section 3. So we may now suppose that $p = 3$. Lemma 2.4 (ii) indicates that $H^* \cong \text{G}_2(3^a)$ with $a \geq 1$, $\text{PSp}_4(3)$, $\text{PSL}_4(3)$, $\text{PSU}_4(3)$,

$\Omega_7(3)$ or $P\Omega_8^+(3)$. The first two possibilities are eliminated by Lemmas 4.1 and 4.2 and the remaining cases are shown to result in the groups listed in Theorem 1.1(iii) in Propositions 4.3, 4.4, 4.14. \square

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